

ON THE GEOMETRY OF SPHERICAL VARIETIES

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ABSTRACT. This is a survey article on the geometry of spherical varieties.

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1. INTRODUCTION

This paper is a survey on the geometry of spherical varieties. The problem of classifying algebraic varieties is very natural. It can be decomposed in two: classify algebraic varieties modulo birational transformations and describe all the elements in a birational class. These two problems are in general extremely hard and it is natural to impose conditions on the varieties we want to classify. The most natural conditions to impose are restrictions on the singularities. It is also natural to impose additional structures. In particular if G is a reductive¹ algebraic group we may consider G -varieties *i.e.* varieties with an action of G . Even with such restrictions the classification problem stays very hard.

The action of G however gives a measure of the complexity of the varieties: the simplest G -varieties being the varieties with finitely many orbits. From this perspective, the simplest class of varieties we can consider is the following class.

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¹Assuming that G is reductive has several implications which make this choice important: finite generation properties, good representation theory.

Definition 1.0.1. A G -spherical variety is a normal G -variety such that all its G -equivariant birational models have finitely many G -orbits.

This definition is not the usual definition of spherical varieties but we want to emphasize this more intrinsic and geometric definition. We will give equivalent definitions in Section 2.1. Spherical varieties generalise several classes of ubiquitous algebraic varieties: toric varieties, projective rational homogeneous spaces and symmetric varieties.

As one can expect from their definition, spherical varieties have a nice equivariant birational behaviour. In particular any projective spherical variety is a Mori Dream Space (see Theorem 4.1.1) and therefore Mori program runs perfectly over any characteristic for spherical varieties (see Section 4.1). Furthermore the above classification program is almost completed: the Luna-Vust theory (see Section 3.1) describes all the birational models of a given spherical variety using colored fans — a generalisations of fans used to classify toric varieties. Furthermore, Luna proved that the description of all birational classes is equivalent to the classification of wonderful varieties and gave a conjectural classification. Two solutions for this classification have been recently proposed (see Section 3.4.4).

Spherical varieties have many other nice geometric properties. The characteristics of the action of the group G imply for example that the Chow groups are equal to the invariant Chow groups and are finitely generated. For smooth spherical varieties, they agree with the homology groups (see Section 2.2). The group action also gives a strong control on the local structure of spherical varieties. In particular all spherical varieties are rational. In characteristic 0, they have rational singularities and therefore are Cohen-Macaulay. This in turn implies vanishing results for nef line bundles (see Section 2.3). For spherical varieties in positive characteristic coming from reduction of spherical varieties in characteristic 0, we have Frobenius splitting properties and the same regularity and vanishing results hold (see Section 4.2).

The classification of spherical varieties in the same birational class via colored fans has many geometric applications. For example, as in the case of toric varieties, there is an explicit description of the Picard group and the group of Weil divisors in terms of convex geometry (see Section 3.2). Among other consequences this yields criteria of \mathbb{Q} -factoriality, quasi-projectivity, properness or affineness for spherical varieties (see Sections 3.1 and 3.2). For projective varieties, the fan geometry can partially be translated into convex properties of the moment polytope associated to the moment mapping (see Subsection 4.3.1). More generally, Okounkov [115] and latter Lazarsfeld and Mustață [98] and Kaveh and Khovanskii [76] defined the so called Okounkov bodies. Many geometric properties of the variety are encoded in the Okounkov body. This takes a particularly simple form for spherical varieties and leads in characteristic 0 to flat deformations of spherical varieties to toric varieties, to a smoothness criterion, to formulas for computing the degree of line bundles and to an explicit presentation of the subring of the cohomology ring generated by the first Chern classes of line bundles (see Section 4.3).

In this paper we tried to concentrate on the geometric properties of spherical varieties and therefore avoided as much as possible considering more particular classes

of spherical varieties such as toric varieties, horospherical varieties or symmetric varieties. We nevertheless discuss the specificities of toroidal and sober varieties in more details.

Toroidal varieties arise naturally from the colored fan classification point of view since by definition their fan do not have colors. For this reason they are tightly connected to toric varieties as their same suggests. Toroidal varieties are also natural from the log-geometric point of view since the smooth toroidal varieties are exactly the log-homogeneous varieties² (see [36], [34] for more on the log-geometric point of view). Furthermore, the study of toroidal varieties leads to the description of a B -stable canonical divisor and to explicit equivariant resolutions of singularities in characteristic 0 of any spherical variety (see Section 3.3).

Sober varieties are the spherical varieties having a complete toroidal variety with a unique closed orbit in their birational class. This completion is unique in the birational class. Thanks to results of Luna [107], the classification of all birational classes of spherical varieties amounts to classifying these complete toroidal varieties with a unique closed orbit which are smooth. Such varieties are called wonderful varieties and a classification has recently been proposed by Cupit-Foutou and by Bravi and Pezzini (see Section 3.4).

Inspired by projective rational homogeneous spaces, the study of closures of B -orbits³ in spherical varieties is an active and rich subject. We give a brief tour and describe in particular a graph whose vertices are the closures of B -orbits. In some special cases the properties of this graph imply normality or non-normality results for orbit closures. We also present an application of these techniques proving regularity results on multiplicity-free subvarieties of projective rational homogeneous spaces (see Section 4.4).

We end the paper with a short list of further examples of spherical varieties.

Notation (groups). In this paper G will be a connected reductive algebraic group. We will denote by H a closed spherical subgroup *i.e.* such that G/H is a spherical variety. We denote by T a maximal torus of G , by W the associated Weyl group and by R the corresponding root system. We denote by B a Borel subgroup of G and by R^+ and R^- the corresponding sets of positive roots and negative roots. We denote by U the unipotent radical of B . For Γ a group, we denote by $\mathfrak{X}(\Gamma)$ the group of characters and by $[\Gamma, \Gamma]$ the derived group. For λ a dominant character of T , we denote by $V(\lambda)$ the highest weight module of highest weight λ . For V a representation of a group Γ , we denote by V^Γ the subspace of invariant and by $V_\lambda^{(\Gamma)}$ the subspace of semiinvariants of weight $\lambda \in \mathfrak{X}(\Gamma)$. The sum of all these subspaces of semiinvariants is denoted by $V^{(\Gamma)}$.

Notation (varieties). We work over an algebraically closed field k . Varieties will be irreducible and reduced. A G -variety is a variety together with an action of the group G . Let H be a spherical subgroup of G . The quotient G/H is called a spherical homogeneous space. Any spherical variety X which is G -birational to G/H is called an embedding of G/H . For Γ an algebraic group and Γ' a closed subgroup acting on a variety X , if the quotient of $\Gamma \times X$ under the action of Γ' given by $\gamma' \cdot (\gamma, x) = (\gamma\gamma', \gamma'^{-1} \cdot x)$ exists, we call it the contracted product and

²For a linear algebraic group.

³For B a Borel subgroup. Spherical varieties have finitely many B -orbits, see Theorem 2.1.2.

write $\Gamma \times^{\Gamma'} X$. Such a quotient exists for example if X has ample Γ' -linearisable line bundle or if the quotient $\Gamma \rightarrow \Gamma/\Gamma'$ is locally trivial for Zarisky topology (e.g. if Γ' is a parabolic subgroup of Γ). The contracted product is a locally trivial fibration (for the étale topology) with fibers isomorphic to X over Γ/Γ' .

Foreword. This is a survey article. I tried to gather the most important results on spherical varieties with an emphasize on the geometry. I also tried to give sketch of proofs of the results when this was not too long as well as to give precise references to all the results presented. The presentation does not always follow a logical order: I sometimes use results coming later in the text but the reader will easily check the consistency of the proofs.

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2. FIRST GEOMETRIC PROPERTIES

2.1. Characterisations of spherical varieties. In this section, we give several equivalent definitions of spherical varieties.

Definition 2.1.1. Let X be a G -variety and V a G -module.

- (i) The complexity $c(X)$ of X is the minimal codimension of a B -orbit in X .
- (ii) V is multiplicity-free if for any $\lambda \in \mathfrak{X}(T)$ dominant, $\dim \operatorname{Hom}_G(V(\lambda), V) \leq 1$.

Theorem 2.1.2. Let X be a normal G -variety. The following are equivalent.

- (i) $k(X)^B = k$.
- (ii) $c(X) = 0$
- (iii) X has finitely many B -orbits.
- (iv) X is spherical.

If X is quasi-projective, then these conditions are equivalent to:

- (v) For \mathcal{L} a G -linearised line bundle, the G -module $H^0(X, \mathcal{L})$ is multiplicity-free.

Proof. (i) \Leftrightarrow (ii) by Rosenlicht's Theorem [137].

(ii) \Rightarrow (iii) Let Y be a closed B -stable subvariety of X we want to prove $c(Y) = 0$. Using the map $P \times^B Y \rightarrow PY$ which is generically finite for P a minimal parabolic subgroup containing B with $PY \neq Y$, we get that $c(PY) = c(P \times^B Y)$ and the latter is easily proved to be greater than $c(Y)$: the fibration $P \times^B Y \rightarrow P/B$ is locally trivial with fiber Y so that $c(P \times^B Y)$ is equal to the minimum of the codimension of the orbits of a subgroup of B in Y . Since the minimal parabolic subgroups containing B span G , we may assume that Y is G -stable. By classical representation theoretic arguments and Sumihiro's local structure Theorem on normal G -varieties (see Theorem 2.3.1), there exists an open B -stable affine subset X_0 meeting Y non trivially such that any element in $k[X_0 \cap Y]^{(B)}$ can be lifted to an element in $k[X_0]^{(B)}$ (up to taking some higher power in positive characteristic, see also [86, Theorem 1.3]). As $f \in k(Y)^B$ can be written u/v with $u, v \in k[X_0 \cap Y]^{(B)}$ with the same weight, there exists $n > 0$ and $u', v' \in k[X_0]^{(B)}$ such that $u'|_Y = u^n$ and $v'|_Y = v^n$. We get $(u'/v')|_Y = f^n$ and the transcendence degree of $k(X)^B$ is bigger than the transcendence degree of $k(Y)^B$.

(iii) \Rightarrow (iv) Follows from (ii) \Rightarrow (iii): $c(\tilde{X}) = 0$ for any G -birational model \tilde{X} .

(uv) \Rightarrow (v) We may assume that $X = G/H$ and that \mathcal{L} is G -linearised. If the multiplicity-free assumption fails, there exist independent vectors v, w in $k[G]^{(H)}$ of the same weight. One can check that the closure of $G \cdot (u + v)$ in $\mathbb{P}(V)$ where V is the G -module spanned by u and v has infinitely many orbits. Taking the closure of $G \cdot (x, u + v)$ for a general $x \in X$ gives a G -birational model with infinitely many orbits.

(v) \Rightarrow (\imath). Let $f \in k(X)^B$, there exists a G -linearised line bundle \mathcal{L} and sections σ, σ' in $H^0(X, \mathcal{L})^{(B)}$ with the same weight such that $f = \sigma/\sigma'$. These sections are colinear and f is constant. \square

Corollary 2.1.3. *Spherical varieties are rational.*

Proof. Let G/H be a spherical homogeneous space, by the previous result, there exists a Borel subgroup B such that $B/B \cap H$ is dense in G/H . It is therefore enough to prove that a quotient B/K is rational for any connected solvable group B and any subgroup scheme K . We proceed by induction on $(\dim B, \dim B/K)$ with the lexicographical order. Pick Z a connected one dimensional normal subgroup of B . Such a subgroup always exists: if the unipotent radical of B is trivial then B is a torus and pick for Z any one dimensional subgroup. Otherwise, pick for Z a one dimensional subgroup of the center of the unipotent radical U of B (this center is non trivial and such a subgroup exist for example by [15, Theorem 10.6(2)]). If $Z \subset K$ then quotienting by Z we conclude by induction on $\dim B$. Otherwise, we have a fibration $B/K \rightarrow B/KZ$ obtained after quotienting by the action of Z via left multiplication (recall that Z is normal and that KZ is the subgroup generated by K and Z). The fiber of this fibration is $Z/(Z \cap K)$ which is a connected group of dimension 1. Since any connected solvable group is special (see [141, Proposition 14]) the fibration $B/K \rightarrow B/KZ$ is locally trivial and since B/KZ is rational (by induction on $\dim B/K$) the result follows. \square

2.1.1. *Comments.* Spherical varieties are usually defined by condition (\imath) or (\imath) of the previous result. Brion [22], Vinberg and Kimel'fel'd [151] and Popov [127] prove the equivalence with condition (\imath). The equivalence with the last condition was proved in [151], [29] and [12]. We refer to [149, Chapter 5] for other equivalent definitions of spherical varieties.

2.2. Chow groups and homology. As a first step into the geometry of spherical varieties, we describe and compare Chow groups, homology groups and cohomology groups of spherical varieties.

For a variety X , we denote by $Z_k X$ the free abelian group generated by all k -dimensional closed subvarieties and by $R_k X$ the subgroup generated by divisor $[\operatorname{div}(f)]$ where $f \in k(W)^\times$ with W a $(k+1)$ -dimensional subvariety in X . If X is a B -variety, we denote by $Z_k^B X$ the free abelian group generated by the B -stable closed subvarieties of X and by $R_k^B X$ the subgroup generated by divisors of the form $[\operatorname{div}(f)]$ where $f \in k(W)^{(B)}$ with W a $(k+1)$ -dimensional subvariety in X . We denote by $A_k X$ and $A_k^B X$ the quotients $Z_k X/R_k X$ and $Z_k^B X/R_k^B X$. We denote by $A_* X$ and $A_*^B X$ the direct sum of the groups $A_k X$ and $A_k^B X$ for all $k \geq 0$.

Theorem 2.2.1. *Let X be a spherical variety, then the canonical homomorphism $A_*^B X \rightarrow A_* X$ is an isomorphism.*

Proof. This result is proved in two steps. First using a result of H. Sumihiro [145] (see also Theorem 2.3.1) there is an equivariant Chow Lemma and by standard exact sequences for Chow groups (see [59, Section 1]) one can reduce the proof to the case where X is projective. This was proved by A. Hirschowitz [69]. The surjectivity of the map comes from the Borel fixed point Theorem applied to the Chow variety: the class in the Chow variety of any subvariety in X has a B -fixed point in the closure of its B -orbit. Applying similar arguments to the rational curves in the Chow varieties proves that $R_k^B X$ generates $R_k X$. \square

Corollary 2.2.2. *Let X be a complete spherical variety.*

- (i) *The groups $A_k X$ are of finite type for all $k \geq 0$.*
- (ii) *The cone of effective cycles in $A_k X \otimes_{\mathbb{Z}} \mathbb{Q}$ is a polyhedral convex cone generated by the classes of the closures of the B -orbits.*
- (iii) *Rational and algebraic equivalences coincide on X .*
- (iv) *Rational and numerical equivalences coincide for Cartier divisors on X .*

Proof. (i) follows from the previous result. (ii) follows from the proof of the previous result: by Chow's Lemma we may assume X projective. The class of an effective cycle has a B -fixed limit in the Chow variety giving the result. For (iii) recall from [57, Section 19.1.4] that the group of algebraically trivial cycles modulo rational equivalence is divisible. By (i) it is finitely generated therefore trivial. Finally (iv) follows from the fact that $\text{Pic}(X)$ is torsion free (see Corollary 3.2.6) and the fact that the group of homologically trivial divisors modulo rational equivalence is a torsion group [57, Section 19.3.3]. \square

Remark 2.2.3. The Chow groups may in general have torsion even for divisors. For example, the affine quadratic cone X in \mathbb{A}^3 over a smooth conic in \mathbb{P}^2 is toric singular with $A_1 X = \mathbb{Z}/2\mathbb{Z}$ (see [60, Example 2.3] for more examples).

When the variety is smooth and complete, we get a stronger result.

Corollary 2.2.4. *If X is a smooth complete spherical variety, then the canonical homomorphism $A_* X \rightarrow H_* X$ is an isomorphism.*

Proof. By easy considerations on the B -orbits, one can prove that for any variety Y with a trivial B -action the equivariant Kunneth map $A_*^B X \otimes A_*^B Y \rightarrow A_*^B(X \times Y)$ is an isomorphism (see [59, Lemma 3]). Using Theorem 2.2.1 we get that the Kunneth map $A_* X \otimes A_* Y \rightarrow A_*(X \times Y)$ is an isomorphism. From there, classical arguments of G. Ellingsrud and S.A. Strømme [54] imply the result: the class of the diagonal $[\Delta] \in A_*(X \times X)$ is in the image of $A_* X \otimes A_* X$. In particular it is of the form $[\Delta] = \sum_i a_i \otimes b_i$ for some $a_i, b_i \in A_* X$ therefore any class c is of the form $c = \sum_i (a_i \cdot c) b_i$. This proves that the classes b_i generate $A_* X$ and $H_* X$ and that numerical and rational equivalences coincide. \square

Remark 2.2.5. If X is not smooth this result fails: there are examples of singular toric varieties with non trivial odd homology groups, see [111] for more details.

2.2.1. Comments. Theorem 2.2.1 and Corollary 2.2.4 are taken from [59]. In that paper the authors prove that the conclusion of Theorem 2.2.1 holds (with almost the same proof) for any scheme X with an action of a connected solvable group B . The first results in this direction were obtained for projective varieties by

Hirschowitz [69]. Corollary 2.2.2 is taken from [28, Section 1.3]. Note that the above results remain true in positive characteristic if we replace singular homology by étale homology. Note also that for X a complete spherical variety, Brion [30] proved that the following equality holds: $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(X), \mathbb{Z}) = A_1(X)$. Recall the definition of the Chow cohomology group $A^1(X)$ from [57, Chapter 17]. The equality $A^1(X) = \mathrm{Hom}_{\mathbb{Z}}(A_1(X), \mathbb{Z})$ also holds, see [59] thus $\mathrm{Pic}(X) = A^1(X)$.

2.3. Local structure of spherical varieties. In this section we describe the local structure of spherical varieties. More precisely, for X a spherical variety, we exhibit open coverings of X with good properties with respect to the action of the group. We deduce regularity properties and cohomology vanishing results for spherical varieties in characteristic 0.

2.3.1. Local structure. Let us first recall a result of Sumihiro [145].

Theorem 2.3.1. *Let X be a normal G -variety and let Y be a G -orbit.*

- (i) *There exists a quasi-projective G -invariant open subset of X containing Y .*
- (ii) *If X is quasi-projective, there exists a G -linearised line bundle \mathcal{L} inducing a G -equivariant embedding in $\mathbb{P}(H^0(X, \mathcal{L})^\vee)$.*

Because of this result, to give a Local Structure Theorem, we may assume that X is quasi-projective equivariantly embedded in $\mathbb{P}(V)$ where V is a rational G -module. Let $\sigma \in V^{\vee(B)}$ a semi-invariant linear form and let P be the stabiliser of $[\sigma] \in \mathbb{P}(V^\vee)$. The group P is a parabolic subgroup containing B . Denote by L the Levi subgroup of P containing T and by P_u the unipotent radical of P . Denote by X_σ the open subset of X where σ does not vanish.

Theorem 2.3.2. (i) *P_u acts properly on X_σ , the quotient X_σ/P_u exists and the morphism $\pi : X_\sigma \rightarrow X_\sigma/P_u$ is affine.*

- (ii) *There is a T -stable closed subvariety Z of X_σ such that the morphisms $P_u \times Z \rightarrow X_\sigma$ and $Z \rightarrow X_\sigma/P_u$ are finite and surjective.*

- (iii) *If $\mathrm{char} k = 0$, then Z can be chosen L -stable and such that morphisms $P_u \times Z \rightarrow X_\sigma$ and $Z \rightarrow X_\sigma/P_u$ are isomorphisms.*

Proof. It is enough to prove the result for $X = \mathbb{P}(V)$. Representation theory of G gives a B -semiinvariant vector $v \in V^{(B)}$ such that $\sigma(v) = 1$. Modifying V we may even assume that for any regular dominant cocharacter $\varpi^\vee : \mathbb{G}_m \rightarrow T$, the weight λ_v of v has value $\langle \varpi^\vee, \lambda_v \rangle$ larger than all the other weights of V . Taking a T -stable complement S of kv in V , we set $Z = \mathbb{P}(S \oplus kv)$. Our assumptions imply that the morphism $P_u \times Z \rightarrow X_\sigma$ is \mathbb{G}_m -equivariant (where \mathbb{G}_m acts via ϖ^\vee), that the only fixed points are $([v], 1)$ and $[v]$, and that the fiber of $[v]$ is $([v], 1)$. This implies that the map is finite and surjective.

Since $P_u \times Z \rightarrow X_\sigma$ is finite, it is proper and we get that $k[Z] = k[Z \times P_u]^{P_u}$ is a finite $k[X_\sigma]$ -module proving the finiteness of the latter. This implies the existence of the quotient. The theorem follows from this. For $\mathrm{char} k = 0$ choose S to be a L -stable complement. \square

Remark 2.3.3. In characteristic 0, the variety Z is L -spherical.

2.3.2. Application to singularities and vanishing. In this subsection we assume $\mathrm{char} k = 0$.

Corollary 2.3.4. *Let X be a spherical variety, then any G -stable subvariety has rational singularities, in particular it is Cohen-Macaulay.*

Proof. By the Local Structure Theorem, we may assume X to be affine. The affine case is proved by deformation see Section 4.3.2 and Corollary 4.3.15. \square

Theorem 2.3.5. *Let $\varphi : X \rightarrow X'$ be a proper G -equivariant morphism between spherical varieties and let \mathcal{L} be a globally generated line bundle on X . Then $R^i \varphi_* \mathcal{L} = 0$ for all $i > 0$.*

Proof. By the Local Structure Theorem, we may assume that X' is affine. Then one restricts to the case where \mathcal{L} is trivial. Indeed, let L^\vee be the total space of the dual line bundle \mathcal{L}^\vee . Since \mathcal{L} is globally generated, we have a proper morphism $\pi : L^\vee \rightarrow Y = \text{Spec}(\oplus_n H^0(X, \mathcal{L}^n))$. Up to replacing Y by its normalisation, these varieties are $G \times \mathbb{G}_m$ -spherical varieties. Since $L^\vee \rightarrow X$ is affine it is enough to prove the result for π and \mathcal{O}_{L^\vee} so that we can assume that $\mathcal{L} = \mathcal{O}_X$. By an application of Sumihiro's Theorem we may assume that the morphism is projective.

We conclude by writing $X = \text{Proj}(A)$ for some normal graded algebra $A = \oplus_n A_n$ such that A is generated by A_0 and A_1 and $X' = \text{Spec}(A_0)$. On $X = \text{Proj}(A)$ we have the line bundle $\mathcal{O}_X(1)$. and a commutative diagram

$$\begin{array}{ccc} Z = \text{Spec}(\oplus_n \mathcal{O}_X(n)) & \xrightarrow{\pi} & \text{Spec} A \\ \psi \downarrow & & \downarrow \\ X & \xrightarrow{\varphi} & X' \end{array}$$

where π is proper and *birational*. The varieties on the top line are spherical for $G \times \mathbb{G}_m$ thus have rational singularities. We get the vanishing $R^i \pi_* \mathcal{O}_Z = 0$ for $i > 0$ and thus the vanishing $H^i(Z, \mathcal{O}_Z) = 0$. The result follows since ψ is affine. \square

Corollary 2.3.6. *Let X be a spherical variety proper over an affine and let \mathcal{L} be a globally generated line bundle, then $H^i(X, \mathcal{L}) = 0$ for $i > 0$.*

The previous result is not true without the assumption “proper over an affine” as the example of $\mathbb{A}^2 \setminus \{0\}$ shows: $H^1(\mathbb{A}^2 \setminus \{0\}, \mathcal{O}) \neq 0$. The next result is a generalisation of both Kodaira and Kawamata-Viehweg vanishings. In the next result, $\kappa(\mathcal{L})$ is the Iitaka dimension of \mathcal{L} i.e. the dimension of $\text{Proj} \oplus_n H^0(X, \mathcal{L}^n)$.

Theorem 2.3.7. *Let \mathcal{L} be a globally generated line bundle on a complete spherical variety X , then $H^i(X, \mathcal{L}^{-1}) = 0$ for $i \neq \kappa(\mathcal{L})$.*

Proof. As in the proof of the previous Theorem, let L^\vee be the total space of the dual line bundle \mathcal{L}^\vee . We have a proper morphism $\pi : L^\vee \rightarrow Y = \text{Spec}(\oplus_n H^0(X, \mathcal{L}^n))$ and up to replacing Y by its normalisation, these varieties are $G \times \mathbb{G}_m$ -spherical varieties. Therefore we have $R^i \pi_* \mathcal{O}_{L^\vee} = 0$ for $i > 0$ and by a result of Kempf [79, Theorem 5] we get $R^{\dim X - \kappa(\mathcal{L})} \pi_* \omega_{L^\vee} = \omega_Y$ while $R^i \pi_* \omega_{L^\vee} = 0$ for $i \neq \dim X - \kappa(\mathcal{L})$. Since $\psi : L^\vee \rightarrow X$ is affine and $\omega_{L^\vee} = \psi^*(\omega_X \otimes \mathcal{L})$, the result follows by projection formula and Serre duality (since X has rational singularities, it is Cohen-Macaulay). \square

Remark 2.3.8. In our setting, Kawamata-Viehweg vanishing Theorem would imply that for \mathcal{L} nef we have $H^i(X, \mathcal{L}^{-1}) = 0$ for $i < \kappa(\mathcal{L})$. Note that on spherical varieties nef and globally generated line bundles agree (see Corollary 3.2.11).

2.3.3. Comments. The Local Structure Theorem was first proved in by Brion, Luna and Vust [40] over a field of characteristic 0. It holds in a weaker form for any G -variety. The positive characteristic version of this result was proved by Knop in [88]. For some specific spherical varieties, the Local Structure Theorem as stated in characteristic 0 holds true in positive characteristic. This is the case for toroidal embeddings of the group G seen as a $G \times G$ -spherical variety (see [144]). This also holds for toroidal embeddings of symmetric varieties in characteristic different from 2 by results of [46]. Some extensions of these results are obtained by Tange in [146].

The regularity and vanishing results of Section 2.3.2 are taken from [25]. We will recover these results via Frobenius splitting techniques in Section 4.2.

3. FANS AND GEOMETRY

3.1. Classification of embeddings. Spherical varieties admit a simple combinatorial description of a given equivariant birational class. This classification of all embeddings extends the classification of toric varieties by fans (see [56], [114]).

Note that there always exists a smallest element in a G -birational class: the dense G -orbit G/H .

Definition 3.1.1. Let G/H a homogeneous spherical variety.

- (i) The set of G -invariant valuations⁴ of $k(G/H)$ is denoted by $\mathcal{V}(G/H)$.
- (ii) The weight lattice of G/H is $\mathbb{X}(G/H) = \{\chi \in \mathfrak{X}(T) \mid k(G/H)_\chi^{(B)} \neq 0\}$. It is a free abelian group of finite rank. Its rank is the rank of the spherical variety. Write $\mathbb{X}^\vee(G/H)$ for the dual lattice and $\mathbb{X}_\mathbb{Q}(G/H)$ and $\mathbb{X}_\mathbb{Q}^\vee(G/H)$ for their tensor product with \mathbb{Q} .
- (iii) The set of colors $\mathcal{D}(G/H)$ is the set of B -stable divisors of G/H .

When there is no possible confusion, we will write \mathbb{X} , \mathbb{X}^\vee , \mathcal{V} and \mathcal{D} for $\mathbb{X}(G/H)$, $\mathbb{X}^\vee(G/H)$, $\mathcal{V}(G/H)$ and $\mathcal{D}(G/H)$. Any valuation ν induces a homomorphism $\rho_\nu : \mathbb{X} \rightarrow \mathbb{Q}$ defined by $\rho_\nu(\lambda) = \nu(f)$ with $f \in k(G/H)_\lambda^{(B)}$. This is well defined since $k(G/H)_\lambda^{(B)}$ is one-dimensional. For any $D \in \mathcal{D}$ there is an associated valuation ν_D . We therefore have a map

$$\rho : \mathcal{V} \cup \mathcal{D} \rightarrow \mathbb{X}_\mathbb{Q}^\vee.$$

It is injective on \mathcal{V} (see [86, Corollary 1.8]) but not on \mathcal{D} in general.

3.1.1. Simple spherical varieties. Let X be an embedding of G/H and let Y be a G -orbit in X . The subset $X_{Y,G} = \{x \in X \mid Y \subset \overline{Gx}\}$ is G -stable, open in X . The G -orbit Y is the unique closed G -orbit of $X_{Y,G}$. In particular X can be covered by G -stable open subsets with a unique closed orbit.

Definition 3.1.2. A spherical variety with a unique closed orbit is called simple.

By the above discussion, spherical varieties are covered by simple spherical varieties. Note also that Sumihiro's Theorem 2.3.1 implies that simple spherical varieties are quasi-projective.

⁴A valuation ν is called invariant if $\nu(g \cdot f) = \nu(f)$ for all $g \in G$ and all $f \in k(G/H)$.

Definition 3.1.3. Let X be a spherical variety and Y be a G -orbit in X .

- (i) Define $\mathcal{D}_Y(X) = \{D \in \mathcal{D} / Y \subset \overline{D}\}$.
- (ii) Define $\mathcal{D}(X) = \cup_Y \mathcal{D}_Y(X)$ the set of colors of X .
- (iii) Define $\mathcal{V}(X)$ the set of G -stable irreducible divisors of X .
- (iv) Define $\mathcal{V}_Y(X) = \{D \in \mathcal{V}(X) / Y \subset \overline{D}\}$.

Theorem 3.1.4. A simple spherical embedding X of G/H with closed orbit Y is completely determined by the pair $(\mathcal{V}_Y(X), \mathcal{D}_Y(X))$.

Proof. First the datum $\mathcal{D}_Y(X)$ determines two canonical B -stable open subsets $X_{Y,B}$ and X_0 of X and G/H respectively by setting

$$X_{Y,B} = X \setminus \bigcup_{D \in \mathcal{D} \setminus \mathcal{D}_Y(X)} \overline{D} \text{ and } X_0 = G/H \setminus \bigcup_{D \in \mathcal{D} \setminus \mathcal{D}_Y(X)} D.$$

In particular X_0 can be recovered from $\mathcal{D}_Y(X)$. Looking at the ring of functions we have $k[X_{Y,B}] = \{f \in k[X_0] / \nu(f) \geq 0 \text{ for all } \nu \in \mathcal{V}_Y(X)\}$. Finally we recover X from $X_{Y,B}$: since X is simple $X = GX_{Y,B}$. \square

Let X be a spherical variety and Y a G -orbit.

Definition 3.1.5. Define $\mathcal{C}_Y^\vee(X) \subset \mathbb{X}_\mathbb{Q}^\vee$ as the cone generated by $\mathcal{V}_Y(X)$ and $\rho(\mathcal{D}_Y(X))$.

Definition 3.1.6. Let G/H be a homogeneous spherical variety.

(i) A colored cone for G/H , is a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subset \mathbb{X}_\mathbb{Q}^\vee$ and $\mathcal{F} \subset \mathcal{D}(G/H)$ having the following properties.

- (CC1) \mathcal{C} is a cone generated by $\rho(\mathcal{F})$ and finitely many elements of \mathcal{V} .
- (CC2) The intersection $\mathcal{C}^\circ \cap \mathcal{V}$ is non empty⁵.

The colored cone $(\mathcal{C}, \mathcal{F})$ is called strictly convex if the following condition holds.

(SCC) The cone \mathcal{C} is strictly convex and $0 \notin \rho(\mathcal{F})$.

- (ii) A colored face of $(\mathcal{C}, \mathcal{F})$ is a pair $(\mathcal{C}_0, \mathcal{F}_0)$ satisfying is the following conditions.
 - (a) The set \mathcal{C}_0 is a face of the cone \mathcal{C} .
 - (b) The intersection $\mathcal{C}_0^\circ \cap \mathcal{V}$ is non empty.
 - (c) We have the equality $\mathcal{F}_0 = \mathcal{F} \cap \mathcal{C}_0$.

Theorem 3.1.7. Let G/H be a homogeneous spherical variety.

(i) The map $X \mapsto (\mathcal{C}_Y^\vee(X), \mathcal{D}_Y(X))$ is a bijection between the isomorphism classes of simple spherical embeddings X of G/H with closed orbit Y and strictly convex colored cones.

(ii) Let X be a spherical embedding of G/H and let Y be an orbit. Then there is a bijection $Z \mapsto (\mathcal{C}_Z^\vee(X), \mathcal{D}_Z(X))$ between the set of G -orbits in X such that $\overline{Z} \supset Y$ and the set of faces of $(\mathcal{C}_Y^\vee(X), \mathcal{D}_Y(X))$.

Proof. The main point in the proof is the existence of a spherical embedding with a given colored cone (uniqueness follows from Theorem 3.1.4). We sketch the construction of a spherical variety X with colored cone $(\mathcal{C}, \mathcal{F})$ and refer to [86, Theorem 3.1] for a complete proof. Condition (CC1) allows to construct elements $(f_i)_{i \in [0, n]} \in k[G]$ which are $B \times H$ -semiinvariants with the same H -character and such that the open subset X_0 defined in the proof of Theorem 3.1.4 is given by $X_0 = D(f_0)$. Write W for the G -span of the $(f_i)_{i \in [0, n]}$. These elements define a

⁵Here \mathcal{C}° denotes the relative interior of \mathcal{C} .

G -equivariant morphism $\varphi : G/H \rightarrow \mathbb{P}(W^\vee)$. Condition (SCC) implies that this morphism has finite fibers and condition (CC2) implies that the Stein factorisation $G/H \rightarrow X \rightarrow \mathbb{P}(W^\vee)$ yields a simple spherical variety X with cone $(\mathcal{C}, \mathcal{F})$. \square

3.1.2. General case. Since spherical varieties have an open covering of simple spherical varieties, we only have to *glue* together the simple pieces to get the general classification. For this we need the following notions.

Definition 3.1.8. A colored fan \mathbb{F} in $\mathbb{X}_{\mathbb{Q}}^\vee$ is a finite collection of colored cones $(\mathcal{C}, \mathcal{F})$ satisfying the properties.

(CF1) Every colored face of a colored cone $(\mathcal{C}, \mathcal{F})$ of \mathbb{F} is in \mathbb{F} .

(CF2) For any $\nu \in \mathcal{V}$ there is at most one colored cone $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}$ with $\nu \in \mathcal{C}^\circ$.

A colored cone is called strictly convex if any colored cone in \mathbb{F} is strictly convex. This is equivalent to $(0, \emptyset) \in \mathbb{F}$.

Definition 3.1.9. Let X be an embedding of G/H we define $\mathbb{F}(X)$ to be the set of all colored cones $(\mathcal{C}_Y^\vee(X), \mathcal{D}_Y(X))$ for Y a G -orbit in X .

Theorem 3.1.10. *The map $X \mapsto \mathbb{F}(X)$ is a bijection between isomorphism classes of embeddings and strictly colored fans.*

Proof. This is a consequence of Theorem 3.1.7. The simple models given by the cones glue together along the smaller simple spherical embedding given by colored faces. One then has to check that the variety obtained this way is separated. This is done thanks to the valuative criterion of separatedness. We refer to [86, Theorem 3.3] for the proof of this fact. \square

3.1.3. Morphisms. Let $\varphi : G/H \rightarrow G/H'$ be a surjective G -morphism of homogeneous spherical varieties. We have a surjection $\varphi_* : \mathbb{X}^\vee(G/H') \longrightarrow \mathbb{X}^\vee(G/H)$ and an equality $\varphi_*(\mathcal{V}(G/H)) = \mathcal{V}(G/H')$.

Definition 3.1.11. (i) Denote by \mathcal{F}_φ the set of $D \in \mathcal{D}(G/H)$ such that φ maps D surjectively to G/H' . Let \mathcal{F}_φ^c be its complement in $\mathcal{D}(G/H)$.

(ii) Denote again by $\varphi_* : \mathcal{F}_\varphi^c \rightarrow \mathcal{D}(G/H')$ the map defined by $D \mapsto \varphi(D)$.

Definition 3.1.12. (i) Let $(\mathcal{C}, \mathcal{F})$ and $(\mathcal{C}', \mathcal{F}')$ be colored cones of G/H and G/H' respectively. We say that $(\mathcal{C}, \mathcal{F})$ maps to $(\mathcal{C}', \mathcal{F}')$ if the following conditions holds.

(CM1) We have the inclusion $\varphi_*(\mathcal{C}) \subset \mathcal{C}'$.

(CM2) We have the inclusion $\varphi_*(\mathcal{F} \setminus \mathcal{F}_\varphi) \subset \mathcal{F}'$.

(ii) Let \mathbb{F} and \mathbb{F}' be colored fans of embeddings of G/H and G/H' respectively. The colored fan \mathbb{F} maps to \mathbb{F}' if every colored cone of \mathbb{F} is mapped to a colored cone of \mathbb{F}' .

Let $\varphi : G/H \rightarrow G/H'$ be a surjective morphism between spherical homogeneous spaces. Let X and X' be embeddings of G/H and G/H' respectively.

Theorem 3.1.13. *φ extends to $X \rightarrow X'$ if and only if $\mathbb{F}(X)$ maps to $\mathbb{F}(X')$.*

Proof. It is enough to assume that X and X' are simple embeddings. Consider the open subsets $X_0, X_{Y,B}$ in X and $X'_0, X'_{Y,B}$ in X' as defined in the proof of Theorem 3.1.4. The condition (CM2) is equivalent to the fact that φ restricts to $\varphi_0 : X_0 \rightarrow X'_0$. Using the characterisation of $X_{Y,B}$ and $X'_{Y,B}$ given in Theorem

3.1.4, condition (CM1) is equivalent to the existence of a morphism $\varphi_{Y,B} : X_{Y,B} \rightarrow X'_{Y,B}$ extending φ_0 . \square

Definition 3.1.14. Let \mathbb{F} be a colored fan. Define the support of \mathbb{F} by

$$\text{Supp}(\mathbb{F}) = \mathcal{V}(G/H) \bigcap \bigcup_{(\mathcal{C}, \mathcal{F}) \in \mathbb{F}} \mathcal{C}.$$

Let $\varphi : X \rightarrow X'$ be a dominant morphism extending a surjective morphism $G/H \rightarrow G'/H'$ between spherical embeddings.

Theorem 3.1.15. φ is proper if and only if $\text{Supp}(\mathbb{F}(X)) = \varphi_*^{-1}(\text{Supp}(\mathbb{F}(X')))$.
In particular X is complete if and only if $\text{Supp}(\mathbb{F}(X)) = \mathcal{V}(G/H)$.

Proof. This is a classical application of the valuative criterion of properness. Note that the inclusion $\text{Supp}(\mathbb{F}(X)) \subset \varphi_*^{-1}(\text{Supp}(\mathbb{F}(X')))$ is always satisfied. \square

Remark 3.1.16. Non strictly convex colored fans classify dominant G -morphisms from G/H with irreducible and reduced fibers, see [86, Theorem 4.5].

3.1.4. *Structure of G -orbits.* Let X be a spherical G -variety. Let us first note the following corollary of Theorems 3.1.7 and 3.1.10.

Corollary 3.1.17. *There is a decreasing bijection between the set of G -orbits in X and the set of colored faces of the fan of X .*

Theorem 3.1.18. *Let Y be a G -orbit, then $\text{rk}(Y) = \text{rk}(X) - \dim \mathcal{C}_Y^\vee(X)$.*

Proof. Let Λ be the set of characters vanishing on $\mathcal{C}_Y^\vee(X)$. Then restriction of B -semiinvariant functions gives the inclusion $\Lambda \subset \mathbb{X}(Y)$. But, up to p -torsion, B -semiinvariant rational functions on Y can be extended to B -semiinvariant rational functions on X (see for example [86, Theorem 1.3b]). \square

Theorem 3.1.19. *Assume $\text{chark} = 0$. Any closed G -stable subvariety Y of X is a spherical variety.*

Proof. Normality follows from Corollary 2.3.4 or Corollary 4.3.15. Theorem 2.1.2 implies that X and thus Y has finitely many B -orbits. Thus Y is spherical. \square

3.1.5. *Comments.* The classification of embeddings, also called *Luna-Vust theory* was first studied in a more general setting by Luna and Vust in [108]. For spherical varieties *i.e.* varieties of complexity 0, this was latter simplified and extended to any characteristic by Knop in [86]. We follow the presentation of Knop [86]. This classification of spherical embeddings can be partially generalised to varieties of complexity 1, see [88], [149], [148] and [147].

To complete the classification of spherical varieties, one needs to classify the homogeneous spherical spaces *i.e.* the spherical subgroups of a reductive group G . Luna [107] conjectured that these subgroups should be classified by combinatorial objects called *spherical systems*. In [103], Losev proves that there is at most one spherical subgroup for each spherical system. There are two recent propositions of a complete proof of the conjecture *i.e.* the existence of a spherical subgroup for any spherical system. The first approach it via combinatoric and follows Luna's proof of type A in [107]. Type D is solved in [18] and type E in [17]. In the papers

[19], [20], [21], Bravi and Pezzini propose a general solution. A more geometric approach via invariant Hilbert schemes as been proposed in [49] by Cupit-Foutou. See Subsection 3.4.4, [37, Section 4.5] and [149, Section 30.11] for more details.

Several partial classifications have been achieved. For example, Akhiezer in [2] classifies the spherical varieties of rank one while Cupit-Foutou [48] classifies the varieties with two orbits. In [139], [140], Ruzzi gives a complete classification of smooth symmetric varieties with Picard number one. In [118], Pasquier classifies the smooth horospherical varieties with Picard number one.

3.2. Divisors. In this section, we describe the Picard group of spherical varieties in terms of their fans. We also describe ample, nef and globally generated line bundles. Comparing with the Weil divisors described in Section 2.2 this leads to a characterisation of \mathbb{Q} -factorial spherical varieties. We also give characterisations of quasi-projective and affine spherical varieties.

3.2.1. The Picard group. Recall from Theorem 2.2.1 that any Weil divisor δ can be written in the form $\delta = \sum_D n_D [D]$ where the sum runs over all B -stable irreducible divisors in X .

Theorem 3.2.1. *The divisor δ is Cartier if and only if for any G -orbit Y there exists $f_Y \in k(X)^{(B)}$ with $n_D = \nu_D(f_Y)$ for any B -stable divisor D containing Y .*

Proof. Assume that δ is Cartier. We may assume that X is simple with closed orbit Y . By Sumihiro's Theorem X is quasi-projective and we may assume that δ is globally generated (any Cartier divisor on a quasi-projective variety can be written as the difference of two globally generated divisors). Since δ is globally generated, there exists a section $\sigma \in H^0(X, \mathcal{O}_X(\delta))^{(B)}$ with $\sigma|_Y \neq 0$ and δ is principal on X_σ .

Conversely, we may again assume that X is simple with closed orbit Y . Replacing δ by $\delta - \text{div}(f_Y)$ we may assume that no component of δ contains Y . We may therefore replace δ by D an irreducible B -stable divisor not containing Y . Denote by $i : X^{\text{reg}} \rightarrow X$ the inclusion of the regular locus. The locus where the sheaf $i_* \mathcal{O}_{X^{\text{reg}}}(D \cap X^{\text{reg}})$ is not locally free is a G -stable closed subvariety (up to taking a finite cover of G so that the sheaf is G -linearised) and contained in D . It is therefore empty (since Y is the only closed G -orbit). \square

Definition 3.2.2. Let X be a spherical variety

- (i) Denote by $\mathcal{C}^\vee(X)$ the union of the cones $\mathcal{C}_Y^\vee(X)$ for Y a G -orbit.
- (ii) Define the set $PL(X)$ of piecewise linear functions as the subgroup of functions l on $\mathcal{C}^\vee(X)$ such that
 - for any G -orbit Y , the restriction l_Y to $\mathcal{C}_Y^\vee(X)$ is the restriction of an element of $\mathbb{X}(G/H)$;
 - for any G -orbit Z with $Z \subset \overline{Y}$, we have $l_Z|_{\mathcal{C}_Y^\vee(X)} = l_Y$.
- (iii) Denote by $L(X)$ the group of linear functions on $\mathbb{F}(X)$.

Remark 3.2.3. The function l only depends on its values on the maximal cones therefore on the cones of the closed orbits in X . Note however that if the maximal cones do not have the maximal dimension then the function l_Y does not determine an element in $\mathbb{X}(G/H)$.

Theorem 3.2.4. *There is an exact sequence*

$$\mathcal{C}^\vee(X)^\perp \cap \mathbb{X} \rightarrow \mathbb{Z}(\mathcal{D} \setminus \mathcal{D}(X)) \rightarrow \text{Pic}(X) \rightarrow PL(X)/L(X) \rightarrow 0.$$

Proof. By Proposition 3.2.1, we have a surjective map $\text{Pic}(X) \rightarrow PL(X)/L(X)$ defined by sending δ to the collection $(f_Y)_Y$. The kernel is given by divisors $\delta = \sum_D n_D [D]$ such that $n_D = 0$ for all $D \in \mathcal{D}_Y(X)$ for some orbit Y thus coming from $\mathbb{Z}(\mathcal{D} \setminus \mathcal{D}(X))$. If a Cartier divisor is principal its image lies in $L(X)$. If f is such that $\text{div}(f)$ is in the kernel, then for all Y and D a B -stable divisor with $D \supset Y$, we have $\langle \rho(\nu_D), f \rangle = 0$ thus $f \in \mathcal{C}^\vee(X)^\perp$. \square

Corollary 3.2.5. *$\text{Pic}(X)$ is of finite rank.*

Corollary 3.2.6. *$\text{Pic}(X)$ is torsion free if there exists a complete G -orbit.*

Proof. The quotient $PL(X)/L(X)$ is free. For Y a complete G -orbit we have $\dim \mathcal{C}_Y^\vee(X) = \dim \mathbb{X}^\vee$ by Theorem 3.1.18 thus $\mathcal{C}^\vee(X)^\perp = 0$. \square

Remark 3.2.7. If X has no complete G -orbit then the Picard group may have torsion as shows the example $X = G$ with G semisimple not simply connected (this is a spherical variety for the action of $G \times G$).

3.2.2. Ample and globally generated line bundles. If X is complete, the function l completely determines for each closed G -orbit Y an element $\chi_Y \in \mathbb{X}(G/H)$ such that $l_Y = \chi_Y$ so that the value $l_Y(\rho(\nu_D)) = \nu_D(\chi_Y)$ is well defined even if $\rho(\nu_D)$ does not lie in $\mathcal{C}_Y^\vee(X)$. This will simplify the statement of the next result for which we will assume that X is complete. For a characterisation of globally generated and ample line bundles in the general setting, see [149, Theorem 17.3].

Definition 3.2.8. A function $l \in PL(X)$ is called convex if $l_Y \leq (l_Z)|_{\mathcal{C}_Y^\vee(X)}$ for any orbit Y and any closed orbit Z . If the inequality is strict, then l is called strictly convex.

Theorem 3.2.9. *Assume X complete and consider a Cartier B -stable divisor*

$$\delta = \sum_{D \in \mathcal{D} \setminus \mathcal{D}(X)} n_D D + \sum_{D \in \mathcal{D}(X) \cup \mathcal{V}(X)} l_Y(\rho(\nu_D)).$$

(i) *The divisor δ is globally generated if and only if l is convex and the inequality $l_Y(\rho(\nu_D)) \leq n_D$ holds for all closed G -orbit Y and all $D \in \mathcal{D} \setminus \mathcal{D}(X)$.*

(ii) *The divisor δ is ample if and only if l is strictly convex and the inequality $l_Y(\rho(\nu_D)) < n_D$ holds for all closed G -orbit Y and all $D \in \mathcal{D} \setminus \mathcal{D}(X)$.*

Proof. (i) We may assume that δ is G -linearised. It is then globally generated if and only for any closed G -orbit Y here exists $\sigma \in H^0(X, \delta)^{(B)}$ such that $\sigma|_Y \neq 0$. This in turn is equivalent to the fact that we can write $\delta = A_Y + \text{div}(f_Y)$ with A_Y effective not containing Y and f_Y a B -semiinvariant function. If χ_Y is the weight of f_Y , then this is equivalent to the inequalities $l(\rho(\nu_D)) \geq \chi_Y(\rho(\nu_D))$ for all $D \in \mathcal{D}(X) \cup \mathcal{V}(X)$ with equality for $D \supset Y$ and $\chi_Y(\rho(\nu_D)) \leq n_D$ for all $D \in \mathcal{D} \setminus \mathcal{D}(X)$.

(ii) By the previous argument, a large multiple of δ separates closed orbits if and only if $l_Y \neq l_Z$ for Y and Z two distinct closed orbits. The proof therefore reduces to the case of a simple spherical variety X with closed orbit Y in which case we may choose $l_Y = 0$. If δ is ample, then $n\delta - \sum_{D \in \mathcal{D} \setminus \mathcal{D}_Y(X)} D$ is globally generated

for large n thus $n_D > 0$ by (i). Conversely, assume $n_D > 0$ for all $D \in \mathcal{D} \setminus \mathcal{D}_Y(X)$ and let η be the canonical section of the effective divisor δ . We have $X_\eta = X_{Y,B}$ which is affine and one easily checks that any $f \in k[X_\eta]$ is of the form σ/η^n for some n and some $\sigma \in H^0(X, n\delta)$. Since $(X_{g \cdot \eta})_{g \in G}$ is a covering, δ is ample. \square

Corollary 3.2.10. *For X spherical and complete, any ample divisor is globally generated.*

Corollary 3.2.11. *For X spherical and complete, nef and globally generated line bundles agree.*

Proof. Recall from Corollary 2.2.2, that rational and numerical equivalence agree for Cartier divisors. Theorem 3.2.9 implies that the cone of globally generated line bundles is the closure of the ample cone. The latter is the nef cone (see [97, Theorem 1.4.23]). \square

3.2.3. Quasi-projective and affine spherical varieties. The description of ample line bundles on spherical varieties leads to the following characterisation of quasi-projective spherical varieties.

Corollary 3.2.12. *A spherical variety X is quasi-projective if and only if there exists a strictly convex \mathbb{Q} -valued function on $\mathcal{C}^\vee(X)$ which is linear on each cone $\mathcal{C}_Y^\vee(X)$.*

In [86], Knop gives a characterisation of affine spherical varieties.

Theorem 3.2.13. *A spherical variety X is affine if and only if X is simple and there exists $\chi \in \mathbb{X}$ such that $\chi|_{\mathcal{V}} \leq 0$, $\chi|_{\mathcal{C}^\vee(X)} = 0$ and $\chi|_{\rho(\mathcal{D} \setminus \mathcal{D}(X))} > 0$.*

3.2.4. A criterion for \mathbb{Q} -factoriality. Comparing the group of Weil divisors with the Picard group gives the following result.

Theorem 3.2.14. *Let X be a spherical variety.*

(i) *The variety X is locally factorial if and only if for any G -orbit Y , the elements $\rho(\nu_D)$ lying in $\mathcal{C}_Y^\vee(X)$ for D any B -stable divisor form a \mathbb{Z} -linearly independent subset of \mathbb{X}^\vee .*

(ii) *The variety X is locally \mathbb{Q} -factorial if and only if for any G -orbit Y , the elements $\rho(\nu_D)$ lying in $\mathcal{C}_Y^\vee(X)$ for D any B -stable divisor form a linearly independent subset of $\mathbb{X}_\mathbb{Q}^\vee$.*

Proof. Since this is local we may assume that X is simple with closed orbit Y . By Theorem 2.2.1, we have an exact sequence $\mathbb{X} \rightarrow \mathbb{Z}(\mathcal{V}(X) \cup \mathcal{D}) \rightarrow A_{\dim X - 1}(X) \rightarrow 0$. Since there is a unique cone in the fan, we have $PL(X) = L(X)$ thus for the Picard group we have an exact sequence $\mathcal{C}_Y^\vee(X)^\perp \cap \mathbb{X} \rightarrow \mathbb{Z}(\mathcal{D} \setminus \mathcal{D}_Y(X)) \rightarrow \text{Pic}(X) \rightarrow 0$, by Theorem 3.2.9. The groups $\text{Pic}(X)$ and $A_{\dim X - 1}(X)$ coincide if and only if for any $D \in \mathcal{V}(X) \cup \mathcal{D}_Y(X)$ there exists $\chi_D \in \mathbb{X}$ with $\chi_D(\rho(\nu_{D'})) = \delta_{D,D'}$ which is the required condition. \square

3.2.5. Comments. The description of the Picard group of spherical varieties follows [24]. See also [149]. The above characterisation of \mathbb{Q} -factorial spherical varieties can be found in [35]. There is no general description of big divisors. For X complete, the cone of effective divisors is convex polyhedral generated by the classes of B -stable divisors (Corollary 2.2.2), but it is hard to give in general explicit generators

of this cone. Partial answers for effective divisors on wonderful varieties or for big and effective divisors on symmetric varieties are given in [33] and [138].

3.3. Toroidal varieties. In this Section we assume $\text{char}(k) = 0$. This section is one of the two sections where we consider a special class of spherical varieties. There are several reasons for this. First, any spherical variety has a toroidal open dense subset whose complement is in codimension at least 2. The computation of a canonical divisor for toroidal varieties therefore leads to a computation of a canonical divisor of any spherical variety. A second reason relies on the fact that the \mathbb{Q} -factoriality criterion (Theorem 3.2.14) turns out to be a smoothness criterion for toroidal varieties. This then leads to an explicit equivariant resolution of singularities for any spherical variety. Finally, using toroidal varieties, we prove that the set of invariant valuations \mathcal{V} is a polyhedral convex cone.

Definition 3.3.1. A spherical variety X is toroidal if $\mathcal{D}(X)$ is empty.

3.3.1. Local structure. Let X be a spherical variety and let $\Delta_X = \cup_{D \in \mathcal{D}} \overline{D}$. Denote by P_X the stabiliser of Δ_X , it is easy to check that P_X is also the stabiliser of BH/H and in particular it is a G -birational invariant of X .

Proposition 3.3.2. *Let X be a spherical variety. The following are equivalent.*

- (i) *The variety X is toroidal.*
- (ii) *There exists a Levi subgroup L of P_X depending only on G/H and a closed subvariety Z of $X \setminus \Delta_X$ stable under L such that the map*

$$(P_X)_u \times Z \rightarrow X \setminus \Delta_X$$

is an isomorphism. The group $[L, L]$ acts trivially on Z which is a toric variety for a quotient of $L/[L, L]$. Furthermore any G -orbit meets Z along a unique L -orbit.

Proof. Assume that X is toroidal, since Δ_X is Cartier and globally generated, we may apply Theorem 2.3.2 to η the canonical section of $\mathcal{O}_X(\Delta_X)$. For some closed L -spherical variety $Z \subset X \setminus \Delta_X$, we have $X_\eta = X \setminus \Delta_X \simeq (P_X)_u \times Z$. Furthermore $B \cap L$ has finitely many orbits in Z and one easily checks that $(BH/H) \cap Z = (G/H) \cap Z$ thus $L = (B \cap L)H$ which implies that $[L, L]$ acts trivially on Z (see [40, Section 3.4, Lemme]). We get that Z is a toric variety for a quotient of $L/[L, L]$.

Conversely, any G -orbit of X meets Z and thus is not contained in Δ_X . It is therefore not contained in any B -stable non G -stable divisor. \square

Remark 3.3.3. The previous result implies that there is a one to one correspondence between the G -orbits in X and the L -orbits in Z : a L -orbit Z' in Z is mapped to GZ' while a G -orbit Y in X is mapped to $Y \cap Z$.

Corollary 3.3.4. *Assume that X is toroidal. Then X is smooth if and only if for any G -orbit Y , the cone $\mathcal{C}_Y^\vee(X)$ is generated by a basis of \mathbb{X}^\vee .*

Proof. The previous result implies that X has the singularities of a toric variety with the same cones as the cones of X . The result follows by standard result on toric varieties (see [56, Section 2.1, Proposition 1] or [114, Theorem 1.10]). \square

3.3.2. Valuation cone. Using toroidal embedding we may prove that the cone \mathcal{V} of invariant valuations is a polyhedral convex cone.

Proposition 3.3.5. *There exists a complete toroidal embedding of G/H .*

Proof. Choose f in $k[G]^{(B \times H)}$ such that f is H -invariant and vanishes on the inverse image in G of any $D \in \mathcal{D}$. Let W be the G -module spanned by f and let X' be the closure of the image of the induced morphism $G/H \rightarrow \mathbb{P}(W^\vee)$. There is no G -orbit of X' contained in the divisor $\text{div}_0(f)$. If X'' is a complete embedding of G/H , then the normalisation X of the closure of G/H diagonally embedded in $X' \times X''$ gives the desired embedding. \square

Corollary 3.3.6. *The set \mathcal{V} is a polyhedral convex cone.*

Proof. First one has to prove that \mathcal{V} is convex. For this result we refer to [117, Proposition 2.1] or [86, Lemma 5.1]. Then pick X a complete toroidal embedding. The union of the finitely many polyhedral convex cones $\mathcal{C}_Y^\vee(X)$ for Y a closed G -orbit is \mathcal{V} . \square

3.3.3. *Resolution of singularities.* As an application we get equivariant resolutions.

Corollary 3.3.7. *Let X be a spherical variety, there exists a G -birational morphism $\tilde{X} \rightarrow X$ with \tilde{X} toroidal.*

Proof. Replace any colored cone $(\mathcal{C}, \mathcal{F})$ by $(\mathcal{C} \cap \mathcal{V}, \emptyset)$. \square

Corollary 3.3.8. *Any spherical variety has a G -equivariant toroidal resolution.*

Proof. Take a G -birational toroidal embedding dominating X . Subdividing its fan as for toric varieties (see [56, Section 2.6] or [114, Section 1.5]) gives a resolution. \square

3.3.4. *Canonical divisor.* Define the canonical sheaf ω_X on a normal variety X by extending the canonical sheaf of the smooth locus: $\omega_X = i_* \omega_{X^{\text{reg}}}$ where $i : X^{\text{reg}} \rightarrow X$ is the embedding of the smooth locus of X . A Weil divisor K_X is called canonical if $\mathcal{O}_X(K_X) = \omega_X$. Denote by ∂X the union of the G -stable divisors.

Theorem 3.3.9. *There exists a canonical divisor K_X of X such that*

$$-K_X = \sum_{X_i \in \mathcal{V}(X)} X_i + \sum_{D \in D(G/H)} a_D \overline{D} = \partial X + H$$

with a_D positive integers and H a globally generated divisor.

Proof. Replacing X by the union of G -orbits of codimension at most one, we may assume that X is toroidal and smooth. We want to study the action map $\text{Act} : \mathfrak{g} \otimes \mathcal{O}_X \rightarrow T_X$ obtained from differentiating the map $G \times X \rightarrow X$ given by the action of G on X . First restrict the tangent bundle to $X \setminus \Delta_X \simeq (P_X)_u \times Z$, we get $T_{X \setminus \Delta_X} \simeq ((\mathfrak{p}_X)_u \otimes \mathcal{O}_{(P_X)_u}) \times T_Z$. Since Z is a toric variety for a quotient S of $L/[L, L]$, if \mathfrak{s} is the Lie algebra of S , we have that the image of the action map for this toric variety is $\mathfrak{s} \otimes \mathcal{O}_Z \simeq T_Z(-\log \partial Z)$ where ∂Z is the union of the S -stable divisors in Z and $T_Z(-\log \partial Z)$ is the logarithmic tangent bundle obtained as subsheaf of T_X of derivations of \mathcal{O}_X preserving the ideal of ∂Z (see [114, Proposition 3.1]). Because of the correspondence between orbits in X and in Z (see Remark 3.3.3) we get that the action map over $X \setminus \Delta_X$ has image $T_{X \setminus \Delta_X}(-\log \partial X)$ which is free. Let $h = \dim H$ and $\mathbb{G}(h, \mathfrak{g})$ be the Grassmann variety of vector subspaces of dimension h in \mathfrak{g} . The kernel of the action map is locally free of rank h and defines a morphism

$X \setminus \Delta_X \rightarrow \mathbb{G}(h, \mathfrak{g})$ which extends the natural morphism $x \mapsto \mathfrak{h}_x$ for $x \in G/H$ and for \mathfrak{h}_x the Lie algebra of the stabiliser of x . By G -equivariance we get a morphism $\varphi : X \rightarrow \mathbb{G}(h, \mathfrak{g})$ and an exact sequence $0 \rightarrow \varphi^* S \rightarrow \mathfrak{g} \otimes \mathcal{O}_X \rightarrow T_X(-\log \partial X) \rightarrow 0$, where S is the tautological subbundle on $\mathbb{G}(h, \mathfrak{g})$. Taking the maximal exterior power we get an isomorphism $\varphi^* \mathcal{O}_{\mathbb{G}(h, \mathfrak{g})}(1) \simeq \omega_X^{-1}(-\partial X)$ and the result follows after picking a well chosen B -stable divisor representing $\mathcal{O}_{\mathbb{G}(h, \mathfrak{g})}(1)$. \square

3.3.5. Rigidity. Toroidal varieties satisfy rigidity properties. It is well known (see for example [83, Theorem 4.6]) that a smooth complex variety X has no deformation if $H^1(X, T_X) = 0$. Write $S_X = T_X(-\log \partial X)$, then the pair $(X, \partial X)$ has no deformation if $H^1(X, S_X) = 0$.

Theorem 3.3.10. *Let X be a smooth spherical variety.*

(i) *If X is toroidal, then $H^i(X, S^* S_X) = 0$ for $i > 0$.*

(ii) *If furthermore G/H is proper over an affine, then $H^i(X, \mathcal{L} \otimes S^* S_X) = 0$ for $i > 0$ and any globally generated line bundles \mathcal{L} .*

Proof. We only give a brief sketch of proof and refer to [90, Theorem 4.1] for the first part and to [13, Proposition 3.2] for the second part.

(i) Let $L_X = \text{Spec } S^* S_X$ and consider the Stein factorisation $L_X \rightarrow M_X \rightarrow \mathfrak{g}^\vee$ of the moment map (see Section 4.3.1 for more on the moment map). Then the hard part is to prove that M_X has rational singularities and that the fiber of $L_X \rightarrow M_X$ is unirational. The main result of [95] finishes the proof.

(ii) The assumption implies the existence of an effective \mathbb{Q} -divisor D containing ∂X such that $\mathcal{O}_{\mathbb{P}(S_X^\vee)}(1) \otimes p^* \mathcal{O}_X(E)$ is big and nef with p the morphism $p : \mathbb{P}(S_X^\vee) \rightarrow X$. Kawamata-Viehweg Theorem (see [78] or [150]) gives the result. \square

For Fano varieties, this result implies a rigidity result (see [13, Proposition 4.2]).

Corollary 3.3.11. *Let X be a smooth toroidal spherical Fano variety, then we have $H^i(X, T_X) = 0$ for all $i > 0$.*

Proof. Write $(X_i)_{i \in [1, n]}$ for the irreducible components of ∂X . If \mathcal{N}_{X_i} is the normal bundle of X_i , then $\mathcal{N}_{X_i} = \omega_X^{-1} \otimes \omega_{X_i}$. The exact sequence $0 \rightarrow S_X \rightarrow T_X \rightarrow \bigoplus_{i=1}^n \mathcal{N}_{X_i} \rightarrow 0$, the above vanishing $H^i(X, S_X) = 0$ and, since ω_X^{-1} is ample, the Kodaira vanishing Theorem $H^1(X_j, \omega_X^{-1} \otimes \omega_{X_i}) = 0$ give the vanishing. \square

3.3.6. Comments. Proposition 3.3.2 is taken from [41]. The description of the set \mathcal{V} as a cone was first proved in [41] and extended to any G -variety in any characteristic in [88]. There is an explicit formula for the coefficients of the canonical divisor given in [30]. The first part of Theorem 3.3.10 is proved in [90] and the second part as well as the rigidity of Fano regular varieties can be found in [13]. These vanishing results were partly extended to quasi-regular varieties in [120]⁶.

3.4. Sober spherical varieties. In this section we assume $\text{char } k = 0$. This is the second section where we consider special classes of spherical varieties. The first reason for this is that sober varieties and especially symmetric varieties and completions of algebraic groups played a leading role in the development of the theory of spherical varieties and still represent an important source of inspiration. The

⁶A quasi-regular variety is a smooth spherical variety such that the components of the boundary are smooth and $\rho(\mathcal{D}(X)) \subset \mathcal{V}$.

second reason is that wonderful varieties *i.e.* the smooth projective toroidal simple sober varieties play a crucial role in the classification of homogeneous spherical varieties.

3.4.1. Structure of the valuation cone and little Weyl group. The set of G -invariant valuations \mathcal{V} of G/H is a convex cone (see Corollary 3.3.6). We give more precise results on its structure. First we characterise the vertex of this cone. Let $\text{Aut}_G(G/H) = N_G(H)/H$ be the G -automorphism group of G/H . We refer to [86, Section 4 and Theorem 6.1] for a proof.

Proposition 3.4.1. *The group $\text{Aut}_G(G/H)$ is a diagonalisable group. In particular, the connected component of the identity $\text{Aut}_G(G/H)^0$ is a central torus.*

Furthermore we have $\dim \text{Aut}_G(G/H) = \dim(\mathcal{V} \cap -\mathcal{V})$.

In particular the cone \mathcal{V} is strictly convex if and only if $N_G(H)/H$ is finite. If this is the case, the subgroup H is called *sober*.

Corollary 3.4.2. *H is sober if and only if G/H has a simple toroidal completion.*

Proof. Take (\mathcal{V}, \emptyset) as colored cone. \square

Definition 3.4.3. An embedding of G/H with H a sober subgroup is called sober.

To describe the structure of the cone of valuations \mathcal{V} we may assume by Proposition 3.4.1 that H is sober. In that case Brion [26] proved the following result.

Theorem 3.4.4. *The cone \mathcal{V} is the fundamental domain of the Weyl group W_X of a root system in \mathbb{X}^\vee . In particular it is simplicial.*

Proof. The main idea of the proof is to use the fact that fundamental domains of the Weyl group of a root system are characterised by the values of the angles between two edges of the cone (which can only be $\pi/2$, $\pi/3$, $\pi/4$ or $\pi/6$). This is proved by restriction to the rank 2 case and Brion proves that for this it is enough to consider spherical varieties of rank at most 4. The proof then follows by case by case inspection.

The root system can be explicitly described as follows. There is a unique simple toroidal completion X of G/H (Corollary 3.4.2) and the Structure Theorem for X (see Proposition 3.3.2) gives an affine toric variety Z for some torus C which is a quotient of T and with $\mathfrak{X}(C) = \mathbb{X}(G/H)$. The generators of the monoid of all weights of elements in $k[Z]^{(C)}$ yields the root system in $\mathfrak{X}(C) = \mathbb{X}(G/H)$. \square

Definition 3.4.5. The group $W_{G/H}$ is called the little Weyl group of G/H .

As the above proof does not give much geometric information or reason why this result should be true, Knop started a project to give a more conceptual definition of the little Weyl group $W_{G/H}$ as well as a more conceptual proof of the fact that \mathcal{V} is fundamental domain for $W_{G/H}$. He achieved this and even generalised most of the statements to any G -variety in a series of papers [88], [89], [92], [85], [90]. See also the text [84]. We will give a very brief review of his results here.

Assume $k = \mathbb{C}$. First using the moment map, Knop defines a little Weyl group in a geometric way and gives many properties of its action. Here we consider the moment map in the complex setting as follows. For X a smooth G -variety let T_X^\vee be the total space of its cotangent bundle, then we have a map

$$\mu : T_X^\vee \rightarrow \mathfrak{g}^\vee$$

defined by $(x, \xi) \mapsto (\eta \mapsto \xi(d_{e,x}\sigma(\eta, 0)))$ where $\sigma : G \times X \rightarrow X$ is the action. We recover the real moment map (see Subsection 4.3.1) from this map since the choice of an invariant hermitian form gives a section of the cotangent bundle. Now we can consider the following fiber product

$$\begin{array}{ccc} T_X^\vee \times_{\mathfrak{t}/W} \mathfrak{t} & \xrightarrow{\quad\quad\quad} & \mathfrak{t} \\ \downarrow & & \downarrow \\ T_X^\vee & \longrightarrow & \mathfrak{g}^\vee \simeq \mathfrak{g} \longrightarrow \mathfrak{g} // G \simeq \mathfrak{t}/W. \end{array}$$

The variety $T_X^\vee \times_{\mathfrak{t}/W} \mathfrak{t}$ is in general not irreducible so pick an irreducible component C dominating T_X^\vee . The Weyl group acts on the irreducible components of $T_X^\vee \times_{\mathfrak{t}/W} \mathfrak{t}$.

Theorem 3.4.6. *The little Weyl group W_X is the quotient $N_W(C)/C_W(C)$. In other words, W_X is the Galois group of the covering $C \rightarrow T_X^\vee$.*

3.4.2. Automorphism free spherical varieties. Recall that sober spherical varieties have a unique simple complete toroidal embedding \bar{X} . The following two problems are natural.

- (1) Give an explicit description of \bar{X} .
- (2) Is \bar{X} smooth?

Definition 3.4.7. A smooth simple complete toroidal embedding is called a wonderful embedding.

There are complete answers to these problems for automorphism free spherical varieties *i.e.* for $H = N_G(H)$. For H satisfying this condition, define the Demazure embedding X_D as the closure of the G -orbit of the Lie algebra \mathfrak{h} of H in the Grassmann variety $\mathbb{G}(h, \mathfrak{g})$ with $h = \dim \mathfrak{h}$. This is a non necessarily normal completion of G/H . Brion [26] proves that the inclusion of G/H in X_D extends to a morphism $\bar{X} \rightarrow X_D$ and that this morphism is the normalisation map. He conjectured that this morphism is an isomorphism and that X_D is smooth. This is now verified.

Theorem 3.4.8. *If $H = N_G(H)$, then the variety \bar{X} is smooth and $\bar{X} = X_D$.*

The first assertion was proved by Knop in [92] and the second by Losev in [102]. Note that for H a spherical subgroup, its normaliser $N_G(H)$ is such that $G/N_G(H)$ is automorphism free (see [149, Lemma 30.2]).

Remark 3.4.9. Knop proves even more: if for all $n \in N_G(H)$ acting trivially on \mathcal{D} we have $n \in H$, then \bar{X} is smooth.

3.4.3. Symmetric varieties. Let G be a reductive group and let θ be a group involution of G . A closed subgroup H of G such that $G^\theta \subset H \subset N_G(G^\theta)$ is called a symmetric subgroup. Vust [152] proved that any symmetric subgroup is spherical.

Definition 3.4.10. An embedding of G/H with H a symmetric subgroup is called a symmetric variety.

The classification of all embeddings is carried out in [153], we review this briefly here. First, let S be a torus in G maximal for the property: $\theta(s) = s^{-1}$ for all $s \in S$. Let T be a maximal torus of G containing S . This torus is stable under θ which induces an involution on the root system R of (G, T) . The restricted root system $R_{G,\theta}$ is defined as

$$R_{G,\theta} = \{\alpha - \theta(\alpha) \mid \alpha \in R\}.$$

This is a root system. Furthermore, there exists a Borel subgroup B containing T such that for $\alpha \in R^+$ we have $\theta(\alpha) = \alpha$ or $\theta(\alpha) \in R^-$. In that case BH/H is dense in G/H and $S_{G,\theta} = \{\alpha - \theta(\alpha) \mid \alpha \text{ simple root}\}$ is a basis of $R_{G,\theta}$. Denote by $S_{G,\theta}^\vee$ the set of simple coroots and by C^\vee the dominant chamber of the dual root system $R_{G,\theta}^\vee$.

Theorem 3.4.11. *Let H with $G^\theta \subset H \subset N_G(G^\theta)$.*

Then $\mathbb{X} = \mathfrak{X}(S/S \cap H)$, $\mathbb{V} = -C^\vee$ and $\rho(\mathcal{D}) = S_{G,\theta}^\vee$. Furthermore the fibers of the map $\rho : \mathcal{D} \rightarrow S_{G,\theta}^\vee$ have at most two elements.

Note that \mathbb{X} , \mathbb{V} and $\rho(\mathcal{D})$ are fixed by G and θ but the set of colors \mathcal{D} depends on the subgroup H .

Remark 3.4.12. The classification of group involutions by Cartan [44] gives a classification of all symmetric subgroups. This yields a complete classification of symmetric varieties via colored cones and restricted root systems. The classification of spherical varieties proposed by Luna in [107] is directly motivated by this classification.

Example 3.4.13. Any reductive group G can be realised as a symmetric $G \times G$ -variety: $G \simeq G \times G/\text{diag}(G)$ where $\text{diag}(G)$ is the diagonal embedding of G . It is the fixed point subgroup of the involution $\theta(x, y) = (y, x)$. In that case the restricted root system is simply the root system of G .

When the group G is adjoint, it is easy to check that the symmetric varieties are automorphism free so that there exists a unique smooth simple toroidal completion. These completions are called *complete symmetric varieties*. The existence of this compactification was first proved in [45]. The complete symmetric varieties have very nice enumerative properties. The most classical example being the variety of complete conics in the cohomology ring of which the Steiner problem can be solved: there are 3264 conics tangent to 5 given conics in general position.

3.4.4. Wonderful varieties and classification. Wonderful varieties are generalisations of complete symmetric varieties.

Definition 3.4.14. A G -variety X is wonderful if the following properties hold.

- X is smooth and projective.
- G has an open orbit Ω in X .
- The boundary $\partial X = X \setminus \Omega$ is a divisor with simple normal crossing. The intersection of all irreducible components $(D_i)_{i \in [1, n]}$ of ∂X is non empty.
- The G -orbit closures are the partial intersections of the divisors $(D_i)_{i \in [1, n]}$.

As the definition suggests, wonderful varieties are well understood from the point of view of log-geometry. In particular wonderful varieties are the complete log-homogeneous varieties with a unique closed orbit. One can also characterise them as the log-homogeneous and log-Fano varieties. We refer to [36],[34] for more on log-homogeneous varieties. Luna [106] proved that wonderful varieties are spherical.

Theorem 3.4.15. *The wonderful varieties are the wonderful embeddings.*

The importance of spherical varieties relies on the fact that their classification is equivalent to the classification of spherical varieties. Indeed, Luna in [107] defines, for H a spherical subgroup its *spherical closure* \bar{H} as the subgroup of elements

in $N_G(H)$ acting trivially on \mathcal{D} . He proves that the classification of the spherical subgroups can be deduced from the classifications of their spherical closure. The advantage is that \bar{H} is sober (since $(N_G(\bar{H})/\bar{H})^\circ$ leaves \mathcal{D} pointwise fixed) and by a result of Knop (see Remark 3.4.9) the simple toroidal completion of G/\bar{H} is smooth therefore it is a wonderful variety.

The main problem is now to determine, for X a wonderful variety which spherical system *i.e.* triples $(\mathbb{X}^\vee(X), \mathcal{V}(X), \mathcal{D}(X))$ actually occur and to describe all the varieties associated to this triple. Losev [102] proved that for each triple, there exists at most one wonderful variety X having the corresponding triple of invariants. In [107], Luna replaces the triple by the triple $(\mathbb{X}^\vee(X), \Sigma(X), \mathcal{D}(X))$ where Σ_X is the set of simple roots generating the cone $\mathcal{V}(X)$ (see Theorem 3.4.4). In the case of symmetric varieties we recover the restricted root system. Luna gives an axiomatic definition of spherical systems and the problem is to construct a wonderful variety associated to each abstract spherical system. Recently two solutions, the first via combinatorics in [19], [20] and [21] and the second via geometry in [49] were proposed.

3.4.5. Comments. Proposition 3.4.1 was first proved in [41, Section 5.2], see [86, Section 4 and Theorem 6.1] for a characteristic free statement. The geometric description of the little Weyl group W_X was first given in [85]. The fact that the cone \mathcal{V} is a fundamental domain for W_X is proved in [89]. Note that the set of invariant valuations forms a cone \mathcal{V} for any G -variety as was proved by Knop in [88] over any algebraically closed fields. Knop conjectures that \mathcal{V} should be the fundamental domain of a Weyl group also in positive characteristic. This question remains open.

4. FURTHER GEOMETRIC PROPERTIES

4.1. Mori theory. There is a general framework for studying the birational geometry of complex projective variety called Mori theory. The minimal model program (MMP) has been proved in dimension at most three with $\text{char } k = 0$ but is still not available in full generality in higher dimension. Large parts of it have been now realised for $\text{char } k = 0$ but very few results are available in general for $\text{char } k > 0$. In this section we will explain that the MMP works perfectly for spherical varieties in all characteristics.

4.1.1. Mori dream spaces. A conceptual reason for the MMP to run perfectly can be stated as follows.

Theorem 4.1.1. *A projective spherical \mathbb{Q} -factorial variety is a Mori dream space.*

Mori dream spaces were introduced by Hu and Keel [71] who proved that for these varieties, the MMP works perfectly in all characteristics (see in particular [71, Proposition 1.11]). Recall a definition of Mori dream spaces⁷.

Definition 4.1.2. (1) Let $(D_i)_{i \in [1, n]}$ be a family of Cartier divisors. Then we write

$$R(X, (D_i)_{i \in [1, n]}) = \bigoplus_{(r_i)_{i \in [1, n]} \in \mathbb{Z}^n} H^0(X, \mathcal{O}_X(\sum_i r_i D_i)).$$

This is a \mathbb{Z}^n -graded ring.

⁷Note that this is not the usual definition of Mori dream spaces, see [71, Definition 1.10 and Theorem 2.9].

(ii) A Mori dream space is a normal projective \mathbb{Q} -factorial variety X with $N^1(X)$ finitely generated and having a basis $([D_i])_{i \in [1, n]}$ of $N^1(X)$ with D_i Cartier for all i and such that the ring $R(X, (D_i)_{i \in [1, n]})$ is finitely generated.

Proof of Theorem 4.1.1. By Corollaries 2.2.2 and 3.2.6 $\text{Pic}(X) = N^1(X)$ is torsion free and finitely generated. Furthermore, the nef cone is a rational polyhedral convex cone (see Theorem 3.2.9). Let $(D_i)_{i \in [1, n]}$ be a set of generators of this group. The following lemma concludes the proof. \square

Lemma 4.1.3. *Let $(D_i)_{i \in [1, n]}$ be a collection of Cartier divisors on a spherical variety X , then $R(X, (D_i)_{i \in [1, n]})$ is finitely generated.*

Proof. Consider the vector bundle V over X associated to the locally free sheaf $\oplus_i \mathcal{O}_X(-D_i)$. The variety V is affine over X and is spherical for the group $G \times \mathbb{G}_m^n$. Since $R(X, (D_i)_{i \in [1, n]}) = H^0(V, \mathcal{O}_V)$ we are left to prove that for X spherical $H^0(X, \mathcal{O}_X)$ is finitely generated.

We only have to prove that $H^0(X, \mathcal{O}_X)^U$ is finitely generated where U is a maximal unipotent subgroup: in characteristic 0, if $H^0(X, \mathcal{O}_X)^U$ is finitely generated, then it generates a G -stable subalgebra R . If A is a complementary G -module then $A^U = 0$ thus $A = 0$. In positive characteristics, see [61, Theorem 9]. But $H^0(X, \mathcal{O}_X)^U$ is generated by $H^0(X, \mathcal{O}_X)^{(B)}$ and since there is a dense B -orbit each eigenspace has dimension at most 1. Let Γ be the monoid of the weights of $H^0(X, \mathcal{O}_X)^{(B)}$, we have to prove that Γ is finitely generated.

Let $\mathbb{X}(\Gamma)$ the subgroup of $\mathfrak{X}(T)$ generated by Γ . Let Ω be the dense B orbit in X . The complement of Ω is a finite union of B -stable divisors $(D)_{D \in \mathcal{D} \cup \mathcal{V}(X)}$. We have $\Gamma = \{\lambda \in \mathbb{X}(\Gamma) / \nu_D(f_\lambda) \geq 0 \text{ for } f_\lambda \in H^0(X, \mathcal{O}_X)^{(B)}_\lambda \text{ and } D \in \mathcal{D} \cup \mathcal{V}(X)\}$. Therefore Γ is a polyhedral convex cone and by Gordan's Lemma it is finitely generated. \square

4.1.2. *Mori program.* The following results, known as contraction Theorem, existence of flips and termination of flips are standard consequences of the above result. We give here sketches of proof in the spherical variety setting.

Recall from Corollary 2.2.2, that the cone of effective curves $NE(X)$ on a projective spherical variety X is convex polyhedral. For $f : X \rightarrow Y$ with X projective, we denote by $NE(X/Y)$ the cone of effective curves contracted by f .

Theorem 4.1.4. *Let X be a projective spherical variety.*

(i) *For each face F of $NE(X)$, there is a unique spherical variety X_F and a unique G -morphism $\text{cont}_F : X \rightarrow X_F$ with connected fibers such that $F = NE(X/X_F)$.*

(ii) *The face F generates the kernel of $(\text{cont}_F)_* : N_1(X)_\mathbb{Q} \rightarrow N_1(X_F)_\mathbb{Q}$ and $N^1(X_F)_\mathbb{Q}$ can be identified to the orthogonal of F in $N^1(X)_\mathbb{Q}$.*

(iii) *Any morphism $\varphi : X \rightarrow Y$ with Y projective such that $F \subset NE(X/Y)$ factors through cont_F .*

Proof. We only construct the contraction and prove the factorisation property. Pick D a Cartier divisor such that D is positive on $NE(X) \setminus F$ and vanishes on F . Then $A = \oplus_n H^0(X, \mathcal{O}_X(nD))$ is a finitely generated normal G -algebra. Define $X_F = \text{Proj}(A)$. This yields the contraction.

If $\varphi : X \rightarrow Y$ satisfies $F \subset NE(X/Y)$, then consider $F' = NE(X/Y)$ and we have that F'/F is a face of $NE(X_F)$. As above there is a morphism $\text{Cont}_{F'/F} :$

$X_F \rightarrow X' = (X_F)_{F'/F}$. Then by unicity we have that $\text{Cont}_{F'} = \text{Cont}_{F'/F} \circ \text{Cont}_F$ and this morphism is also the Stein factorisation of φ concluding the proof. \square

Theorem 4.1.5. *Let X be a \mathbb{Q} -factorial projective spherical variety and let R be an extremal ray of $NE(X)$ such that $\text{Cont}_R : X \rightarrow X_R$ is an isomorphism in codimension one.*

Then there exists a unique \mathbb{Q} -factorial projective spherical variety X^+ and a unique birational morphism $\varphi^+ : X^+ \rightarrow X_R$ called the flip of Cont_R such that

- $\varphi^+ = \text{Cont}_{R^+}$ is the contraction of an extremal ray R^+ in $NE(X^+)$.
- φ^+ is an isomorphism in codimension one.
- The spaces $N^1(X)$ and $N^1(X^+)$ are identified via $\varphi^+ \circ \text{Cont}_R^{-1}$ and the half-lines R and R^+ are opposite in $N_1(X)$.

Proof. We will only construct the morphism $\varphi^+ : X^+ \rightarrow X_R$. Let C be a curve in X such that $[C]$ spans R and let D be a Cartier divisor with $D \cdot C < 0$. Then $A = \bigoplus_n (\text{Cont}_R)_* \mathcal{O}_X(nD)$ is finitely generated (use the Local Structure Theorem to pass from the relative setting to the global setting and use Lemma 4.1.3) and up to taking some power of D it is a normal G -algebra. Define $X^+ = \text{Proj}(A)$. This yields the flip. \square

Let D be a divisor and let $\text{Cont}_{R^+} : X^+ \rightarrow X_R$ be the flip of $\text{Cont}_R : X \rightarrow X_R$. We say that the flip is D -directed if $D \cdot R < 0$ and $D \cdot R^+ > 0$.

Theorem 4.1.6. *There is no infinite sequence of D -directed flips.*

Proof. First remark that X^+ and X are birational and isomorphic in codimension one. In particular they share the same sets of B -stable and G -stable divisors. Having these two sets fixed, there are only finitely many possible fans for X^+ and by Theorem 3.1.10 this implies that there are only finitely many possible embeddings of G/H obtained by flips from X . In particular, there exists a \mathbb{Q} -factorial embedding \tilde{X} of G/H which dominates all the flips obtained from X . Let X^+ be one of these flips, then we have morphisms $f : \tilde{X} \rightarrow X$ and $f^+ : \tilde{X} \rightarrow X^+$ and we claim that

$$D_{\tilde{X}} = f^*D + \sum_i a_i E_i = f^{+*}D^+ + \sum_i a_i^+ E_i$$

with $a_i^+ \geq a_i$ and inequality for some index i (here D^+ is the strict transform of D). This will conclude the proof. To prove this assertion remark that $f^{+*}D^+ - f^*D$ is nef relatively to the composition $\tilde{X} \rightarrow X \rightarrow X_R$ and since any relatively nef divisor is globally generated (this is a relative version of Corollary 3.2.11) the result follows. \square

4.1.3. Comments. The results of this section are taken from [28] and [38]. The proof of Lemma 4.1.3 is taken from [87]. Most of these results can be stated in a more general context: all the results remain true for the relative minimal model program over a G -spherical variety S . Furthermore, these statements are true for varieties of complexity 1. See [38] for these generalisations. In [28], a more precise study of the cone of curves as well as the description of part of the above results in terms of colored fans (see Section 3.1) has been achieved.

4.2. Frobenius splitting. In this section we prove that starting with a spherical variety in characteristic 0, its reductions modulo a prime p are Frobenius split for almost all p . We deduce different proofs of Corollary 2.3.4 and Theorem 2.3.5.

4.2.1. *Existence of a splitting.* For a variety X defined over an algebraic closed field k of positive characteristic p , we say that X is Frobenius split if the *absolute* Frobenius morphism $F : X \rightarrow X$ has a section on the level of structural sheaves *i.e.* the morphism $\mathcal{O}_X \rightarrow F_*\mathcal{O}_X$ has a section $\varphi : F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$. A subvariety $Y \subset X$ is compatibly split for φ if its sheaf of ideals \mathcal{I}_Y satisfies $\varphi(F_*\mathcal{I}_Y) \subset \mathcal{I}_Y$. We will use the book [39] for further reference on Frobenius splittings.

Let X be spherical and defined over an algebraically closed field of characteristic 0 and write X_p for its reduction to an algebraically closed field of characteristic p .

Theorem 4.2.1. *The variety X_p is Frobenius split with splitting φ for all but finitely many p and φ compatibly splits all the closed G -stable subvarieties.*

Proof. Since any spherical variety has a completion which itself has a resolution by a complete toroidal variety, we may assume X smooth complete and toroidal (using [39, Lemma 1.1.7 and 1.1.8]). In particular X_p is smooth for all except finitely many primes. Since any closed G -variety is the intersection of G -stable divisors, it is enough to prove that X is split by a $(p-1)$ -power of a section of ω_X compatibly splitting all G -stable divisors (see [39, Proposition 1.2.1 and Theorem 1.4.10]).

Theorem 3.3.9 implies that for Y a closed orbit in X , we have $\omega_Y^{-1} = H|_Y$ with H globally generated. Since Y_p is projective homogeneous there exists $\sigma \in H^0(Y_p, \omega_{Y_p}^{-1})$ such that σ^{p-1} splits Y . But in characteristic zero, $H^0(Y, \omega_Y^{-1})$ is irreducible and H globally generated therefore the map $H^0(X, H) \rightarrow H^0(Y, \omega_Y^{-1})$ is surjective. This is still true for X_p, Y_p for p large enough thus there is a lift τ of σ . Multiply τ by the canonical section of ∂X and take a $(p-1)$ -power to get the desired splitting. \square

Corollary 4.2.2. *Let X be a spherical variety and Y a closed G -stable subvariety, then for all but finitely many p the variety Y_p has rational singularities.*

Proof. It is enough to prove that X_p has rational singularities for large p and that Y_p is normal for large p since the Y_p is again a spherical variety. We may assume X affine and consider a toroidal resolution $\pi : \tilde{X} \rightarrow X$. Pick a relative ample B -stable divisor D . By Theorem 3.3.9, we have $(1-p)K_{\tilde{X}_p} \geq D$ for p large. In particular the splitting which compatibly splits the closed G -stable subvarieties is a D -splitting by [39, Remark 1.4.2]. Furthermore since π is G -equivariant, the splitting vanishes on the exceptional locus. By [39, Theorem 1.2.8 and 1.3.14], we get $H^0(X_p, R^i\pi_*\mathcal{O}_{\tilde{X}_p}) = H^i(\tilde{X}_p, \mathcal{O}_{\tilde{X}_p}) = 0$ for $i > 0$ thus $R^i\pi_*\mathcal{O}_{\tilde{X}_p} = 0$ and $R^i\pi_*\omega_{\tilde{X}_p} = 0$ for $i > 0$. This proves that X has rational singularities. We also get, for \tilde{Y} a closed G -stable subvariety of \tilde{X} , surjections $H^0(\tilde{X}, \pi^*\mathcal{L}) \rightarrow H^0(\tilde{Y}, \pi^*\mathcal{L}|_{\tilde{Y}})$ for \mathcal{L} globally generated on X . By [39, Lemma 3.3.3] this proves $\pi_*\mathcal{O}_{\tilde{Y}} = \mathcal{O}_Y$ for $Y = \pi(\tilde{Y})$ proving the normality of the closed G -stable subvarieties in X . \square

Corollary 4.2.3. *Assume that X is proper over an affine, then for all but finitely many p and for \mathcal{L} nef on X , we have $H^i(X_p, \mathcal{L}_p) = 0$ for $i > 0$ and the map $H^0(X_p, \mathcal{L}_p) \rightarrow H^0(Y_p, \mathcal{L}_p)$ is surjective for all Y closed and G -stable in X .*

Proof. Pick a toroidal resolution $\pi : \tilde{X} \rightarrow X$. By the previous result we have $\pi_*\mathcal{O}_{\tilde{X}_p} = \mathcal{O}_{X_p}$ and $R^i\pi_*\mathcal{O}_{\tilde{X}_p} = 0$ for $i > 0$ so the results follow from the toroidal case explained in the proof of the previous Corollary. \square

4.2.2. Comments. Frobenius splitting techniques were first introduced by Mehta and Ramanathan [130], [112] to study Schubert varieties. For some special classes of spherical varieties, more precise results have been obtained. Any toric variety is Frobenius split in any characteristic (see for example [121]). The spherical $G \times G$ -embeddings of a reductive group G are also Frobenius split in any characteristic and it can be proved that they admit a rational resolution so that they are Cohen-Macaulay (see [144] or [39, Chapter 6]). In [146], R. Tange explains how to extend these splitting results to some other cases using generalised parabolic induction techniques. More generally, in any characteristic different from 2, any embedding of a symmetric variety (see Definition 3.4.10) is Frobenius split compatibly with the closed G -stable subvarieties (this was proved in [46] for the Wonderful compactification, it is easily extended to all embeddings, see [125]). This was used in [125] to study the Gauß map in positive characteristic for cominuscule homogeneous spaces.

Some much more precise splitting results have been obtained by X. He and J.F. Thomsen in [66] and [67]. In particular they applied these results to describe the singularities of $B \times B$ -orbit closures in spherical $G \times G$ -embeddings of a reductive group G (see also Subsection 4.4.4).

4.3. Convex geometry. In this section we assume $\text{char } k = 0$ and X to be a projective. We associate to X several convex polytopes and see that many geometric properties of X can be described in terms of these polytopes.

We prove that the moment map gives rise to another characterisation of spherical varieties and that the classification by colored cones can partially be translated into convex geometry of the image of the moment map. Using more general convex polytopes we construct flat deformation to toric varieties, give a smoothness criterion and describe a subring of the cohomology ring of spherical varieties.

4.3.1. Moment map. Assume $k = \mathbb{C}$. For V a G -module and X a projective G -variety equivariantly embedded in $\mathbb{P}(V)$, there exists a moment map $\mu : X \rightarrow \mathfrak{k}^\vee$ where \mathfrak{k} is the Lie algebra of a maximal compact subgroup K of G . The map can be defined as follows. Let $\langle \cdot, \cdot \rangle$ be an hermitian K -invariant scalar product on V , then $\mu(x)(\kappa) = i \frac{\langle x, \kappa \cdot x \rangle}{\langle x, x \rangle}$. If T is a maximal torus of G such that $T_K = T \cap K$ is a maximal torus in K , then the dual \mathfrak{k}_K^\vee of the Lie algebra of T_K is the subspace of T_K -invariants in \mathfrak{k}^\vee . This subspace contains the dominant chamber $\mathfrak{k}_{K,+}^\vee$. The following is a classical result.

Theorem 4.3.1. $\mu(X) \cap \mathfrak{k}_{K,+}^\vee$ is convex.

In this context, Brion gave a new proof of this result by reinterpreting the convex polytope $\mu(X) \cap \mathfrak{k}_{K,+}^\vee$. We shall see that the polytope contains many information on the geometry of the variety X . Let \mathcal{L} be an ample G -linearised line bundle on X and let $R(X, \mathcal{L}) = \bigoplus_n H^0(X, \mathcal{L}^{\otimes n})$.

Definition 4.3.2. Set $\Delta(X, \mathcal{L}) = \{\lambda \mid \exists n : n\lambda \in \mathfrak{X}(T) \text{ and } H^0(X, \mathcal{L}^{\otimes n})_{n\lambda}^{(B)} \neq 0\}$.

Because the algebra of U -invariants $R(X, \mathcal{L})^U$ is integral of finite type, the above set $\Delta(X, \mathcal{L})$ is a convex polytope (see for example [23, Proposition 2.1]). Furthermore one easily checks that it spans an affine space with vector space direction $\mathbb{X}_{\mathbb{Q}}$ in $\mathfrak{X}(T)_{\mathbb{Q}}$ (see [35, Proposition 1.2.3]). Assume that $\mathcal{L} = \mathcal{O}_{\mathbb{P}(V)}(1)$.

Proposition 4.3.3. We have $\Delta(X, \mathcal{L}) = \mu(X) \cap \mathfrak{k}_{K,+}^\vee \cap \mathfrak{X}(T)_{\mathbb{Q}}$.

Proof. The main idea is that for λ dominant, results of Mumford (see [113, Appendix]) imply that the weight λ/n lies in $\mu(X)$ if and only if $X \times G/P_\lambda$ has a stable point with respect to the polarisation $\mathcal{L}^{\otimes n} \otimes \mathcal{O}_{G/P_\lambda}(1)$ (here P_λ is the parabolic subgroup stabilising λ). This in turn is equivalent to the existence of a non-trivial invariant and therefore a non trivial map $V_{m\lambda} \rightarrow H^0(X, \mathcal{L}^{\otimes mn})$ proving the result. \square

We now characterise spherical varieties thanks to the moment map.

Proposition 4.3.4. *Let X be a normal G -variety, then X is spherical if and only if for each $x \in X$, the fibre $\mu^{-1}(\mu(x))$ is an orbit under the stabiliser $K_{\mu(x)}$ of $\mu(x)$.*

Proof. We keep the notation of the previous proof. A result of Kirwan [82] implies: for λ dominant, with $\lambda/n \in \mu(X)$, the quotient $\mu^{-1}(-\lambda/n)/K_{-\lambda/n}$ is homeomorphic to the GIT quotient $(X \times G/P_\lambda)//G$ (with polarisation $\mathcal{L}^{\otimes n} \otimes \mathcal{O}_{G/P_\lambda}(1)$ as in the proof of Proposition 4.3.3). In particular this quotient is connected. If X is spherical, then the GIT quotient is finite and thus reduced to a point. Conversely, if the quotient of the fiber is a point, so is the GIT quotient thus the algebra $R(X \times G/P_\lambda, \mathcal{L}^{\otimes n} \otimes \mathcal{O}_{G/P_\lambda}(1))^G$ is of dimension 1. Its degree 1 piece $\text{Hom}(V_\lambda, H^0(X, \mathcal{L}^{\otimes n}))^G$ is thus of dimension at most 1 proving that X is spherical by Theorem 2.1.2. \square

We describe the connection between the colored fan of X and the convex polytope $\Delta(X, \mathcal{L})$. Let $\sigma \in H^0(X, \mathcal{L})^{(B)}$, we can write

$$\text{div}(\sigma) = \sum_{D \in \mathcal{V}(X) \cup \mathcal{D}} n_D D,$$

with $n_D \geq 0$. Let λ_σ be the B -weight of σ .

Proposition 4.3.5. $\Delta(X, \mathcal{L}) = \{\lambda_\sigma + \xi \mid \langle \rho(\nu_D), \xi \rangle + n_D \geq 0 \text{ for } D \in \mathcal{V}(X) \cup \mathcal{D}\}$

Proof. This comes from the fact that a section of $H^0(X, \mathcal{L}^{\otimes n})$ is of the form $\sigma^n f$ for some $f \in k(X)$ and defines an effective divisor. \square

Some of the fan geometry can be translated into convex geometry of the polytope $\Delta(X, \mathcal{L})$. We only state the following result.

Theorem 4.3.6. *Let Y be a G -orbit Y .*

- (i) $\Delta(\overline{Y}, \mathcal{L})$ is a face of $\Delta(X, \mathcal{L})$.
- (ii) $\mathcal{C}_Y^\vee(X)$ is the dual cone: $\mathcal{C}_Y^\vee(X) = \{\nu \in \mathbb{X}_\mathbb{Q}^\vee \mid \langle \nu, \xi \rangle \geq 0 \ \forall \xi \in \Delta(\overline{Y}, \mathcal{L})\}$.
- (iii) $\mathcal{D}_Y(X) = \{D \in \mathcal{D} \mid \langle \rho(\nu_D), \xi - \lambda_\sigma \rangle + n_D = 0, \ \forall \xi \in \Delta(\overline{Y}, \mathcal{L})\}$.
- (iv) The map $Y \mapsto \Delta(\overline{Y}, \mathcal{L})$ is a bijection from the G -orbits onto the set of faces of $\Delta(X, \mathcal{L})$ whose interior meets \mathcal{V} .

4.3.2. Deformation to toric varieties. There is a general framework steaming from convex geometry which leads to the construction of flat deformation of projective varieties to toric varieties. This was first initiated by Okounkov [115] and [116] and developed in several directions (see [98], [5]). For spherical varieties, this was first developed by Kaveh [73] and Alexeev and Brion [3] inspired in many aspects by the work of Caldero [43] on toric degeneration of projective rational homogeneous spaces.

Definition 4.3.7. Let Γ be a semigroup in $\mathbb{N} \times \mathbb{Z}^d$, we define the cone $C(\Gamma)$ as the cone in $\mathbb{R} \times \mathbb{R}^d$ generated by Γ and the Okounkov body as $\Delta(\Gamma) = C(\Gamma) \cap (1 \times \mathbb{R}^d)$.

Choose a total ordering which respects addition on \mathbb{Z}^d , a \mathbb{Z}^d -valuation on a field K is a map $\nu : K \setminus \{0\} \rightarrow \mathbb{Z}^d$ such that $\nu(f + g) \geq \min(\nu(f), \nu(g))$ and $\nu(fg) = \nu(f) + \nu(g)$.

Example 4.3.8. Let X be a projective variety of dimension n and $(Y_i)_{i \in [1, n]}$ a complete flag of subvarieties with Y_i normal of dimension i , then we may define a valuation $\nu_{Y_\bullet} : k(X) \rightarrow \mathbb{Z}^n$ by the order of functions on the subvarieties Y_i : $\nu_{Y_\bullet}(f) = (\text{ord}_{Y_1}(f), \dots, \text{ord}_{Y_n}(f))$.

Example 4.3.9. Starting with a line bundle \mathcal{L} on X and ν a \mathbb{Z}^d -valuation on $k(X)$, we may define the corresponding Newton-Okounkov body. Define the semigroup

$$\Gamma = \Gamma_\nu(X, \mathcal{L}) = \{(k, \nu(f)) \in \mathbb{N} \times \mathbb{Z}^d \mid \text{for } f \in H^0(X, \mathcal{L}^{\otimes k})\}.$$

The cone $C(\Gamma)$ and the Okounkov body $\Delta(\Gamma)$ will be in this case denoted by $C_\nu(X, \mathcal{L})$ and $\Delta_\nu(X, \mathcal{L})$.

Example 4.3.10. The moment body $\Delta(X, \mathcal{L})$ is an Okounkov body for the semigroup obtained using the weights of B -semiinvariant functions by setting

$$\Gamma(X, \mathcal{L}) = \{(k, \lambda) \in \mathbb{N} \times \mathfrak{X}(T) \mid H^0(X, \mathcal{L}^{\otimes k})_\lambda^{(B)} \neq 0\}.$$

Let Γ be a semigroup in $\mathbb{N} \times \mathbb{Z}^d$ and let $R = \bigoplus_n R_n$ be a graded ring with a filtration $(\mathcal{F}_\gamma)_{\gamma \in \Gamma}$ indexed by Γ satisfying the following conditions:

- for $\gamma \leq \gamma'$, we have $\mathcal{F}_\gamma \subset \mathcal{F}_{\gamma'}$
- $\mathcal{F}_\gamma \cdot \mathcal{F}_{\gamma'} \subset \mathcal{F}_{\gamma + \gamma'}$.

Denote by $\text{Gr}_{\mathcal{F}}(R)$ the graded algebra associated to the filtration. Denote by $\mathcal{F}_{\leq \gamma}$ (resp. $\mathcal{F}_{< \gamma}$) the union of the $\mathcal{F}_{\gamma'}$ with $\gamma' \leq \gamma$ (resp. $\gamma' < \gamma$). We say that the filtration has leaves of dimension one if $\dim \mathcal{F}_{\leq \gamma} / \mathcal{F}_{< \gamma} \leq 1$.

Example 4.3.11. Starting with a valuation coming from a flag of subvarieties as in Example 4.3.8, then the filtration obtained from the lexicographical order on \mathbb{Z}^n by $\mathcal{F}_\gamma = \{f \mid \nu(f) \geq \gamma\}$ has leaves of dimension one (see [98, Lemma 1.3]).

Theorem 4.3.12. Assume that Γ is finitely generated.

(i) Then there exists a finitely generated subalgebra $\mathcal{R} \subset R[t]$ such that

- \mathcal{R} is flat over $k[t]$;
- $\mathcal{R}[t^{-1}] \simeq R[t, t^{-1}]$;
- $\mathcal{R}/t\mathcal{R} \simeq \text{Gr}_{\mathcal{F}}(R)$.

(ii) If furthermore the filtration has leaves of dimension one and if Γ contains all the rational points $(\mathbb{N} \times \mathbb{Z}^d) \cap C(\Gamma)$ of the cone $C(\Gamma)$, then the limit $\text{Proj}(\text{Gr}_{\mathcal{F}}(R))$ is normal and is the toric variety associated to the convex polytope $\Delta(\Gamma)$.

Proof. (i) This proof is now classical, we follow [43] and [3]. The ring $\text{Gr}_{\mathcal{F}}(R)$ is finitely generated. Pick generators (\bar{f}_i) of degree γ_i lift them in (f_i) in generators of R . Consider the morphism of graded rings $S = k[x_i] \rightarrow R$, $x_i \mapsto f_i$. Pick generators \bar{g}_k of degree ν_k of the kernel. We have $\bar{g}_k(f_i) \in R_{< \nu_k}$ and we may find $g_k \in \bar{g}_k + S_{< \nu_k}$ with $g_k(f_i) = 0$. Since the associated graded map is an isomorphism, so is the map $S/(g_k) \rightarrow R$. Pick a linear map $\mathbb{Z} \times \mathbb{Z}^d \rightarrow \mathbb{Z}$ such that π is positive on $\gamma_i - \nu_k$ and the generators of $\mathbb{Z} \times \mathbb{Z}^d$ (for the existence see [5, Lemma 5.2]). Define $R_{\leq j}$ as the span of the monomials in f_i whose degree has value at most j under the map π . Then the ring $\mathcal{R} = \bigoplus_n R_{\leq j} t^j$ satisfies the conditions.

(ii) The filtration being with leaves of dimension one, the limit algebra $\text{Gr}_{\mathcal{F}}(R)$ is the algebra $k[\Gamma]$ of the semigroup Γ . Furthermore, the condition on the semigroup is classically equivalent to the normality (see [56, Section 2.1 Proposition 2] or [114, Proposition 1.2]). \square

We may now apply this result to spherical varieties. This is based on two main facts. First the multiplicity-free property of spherical varieties, second combinatorial properties of dual canonical bases.

Let X be a projective spherical variety and let \mathcal{L} be an ample G -linearised line bundle. Let $R = R(X, \mathcal{L}) = \bigoplus_n H^0(X, \mathcal{L}^{\otimes n})$. This ring is multiplicity-free *i.e.*

$$(1) \quad R_n = \bigoplus_{(n, \lambda) \in \Gamma(X, \mathcal{L})} V(\lambda),$$

with $V(\lambda)$ the representation of highest weight λ and $\Gamma(X, \mathcal{L})$, the semigroup defined in Example 4.3.10.

The dual canonical basis $(v_{\lambda, \phi})$ is a basis of $V(\lambda)$ for all λ . Furthermore there exists a so called *string* parametrisation by elements in \mathbb{Z}^d (with $d = \dim U$) satisfying the following conditions (see [101]). Let $\Gamma_{can} = \{(\lambda, \phi) \in \mathfrak{X}(T) \times \mathbb{Z}^d \mid \exists v_{\lambda, \phi} \neq 0\}$.

Theorem 4.3.13. Γ_{can} is the intersection of a polyhedral cone \mathcal{C}_{can} with $\mathfrak{X}(T) \times \mathbb{Z}^d$.

Let Δ_{can} be the Okounkov body of Γ_{can} . The multiplicative properties of the dual canonical basis (proved in [43]) imply that the set

$$\Gamma_{str} = \{(n, \lambda, \phi) \in \mathbb{N} \times \mathfrak{X}(T) \times \mathbb{Z}^d \mid (n, \lambda) \in \Gamma(X, \mathcal{L}), (\lambda, \phi) \in \Gamma_{can}\}$$

is a semigroup called the string semigroup of R . We have an isomorphism of graded module

$$(2) \quad R \simeq k[\Gamma_{str}]$$

and the previous results easily imply.

Corollary 4.3.14. Let X be a projective spherical variety with \mathcal{L} ample, then there exists a flat deformation to a toric variety $X_0 = \text{Proj} k[\Gamma_{str}]$

Corollary 4.3.15. Projective⁸ spherical varieties have rational singularities.

Proof. Since toric varieties have rational singularities (see [56, Section 3.5 last Proposition], [114, Corollary 3.9]), this follows from the stability of rational singularities under deformations (see [53]). \square

4.3.3. Okounkov bodies and degree of line bundles. After Okounkov [115] and Lazarsfeld and Mustařă [98], Kaveh and Khovanskii study, in a series of papers [76], [77], the relationship between the geometry of varieties (in particular the growth of sections of line bundles) with the convex geometry of some semigroups in $\mathbb{N} \times \mathbb{Z}^d$. In particular they prove the following approximation result (see [76, Theorem 1.13]).

Theorem 4.3.16. Let Γ be a semigroup in $\mathbb{N} \times \mathbb{Z}^d$, let Δ be the corresponding Okounkov body and let $H_{\Gamma}(k)$ be the number of elements in $\Gamma \cap (\{k\} \times \mathbb{Z}^d)$.

(i) Then $H_{\Gamma}(k) \sim \text{vol}(\Delta)k^q$ where $q = \dim \Delta$ and $\text{vol}(\Delta)$ is the volume of the convex body Δ^9 .

⁸This statement is also true for affine spherical varieties. Indeed, in the affine case, the same arguments lead to the existence of a deformation to an affine normal toric variety.

⁹The volume is normalised so that the volume of the quotient $\mathbb{R} \times \mathbb{R}^d / \mathbb{Z} \times \mathbb{Z}^d$ is 1.

(u) Let F be a polynomial function of degree n on \mathbb{R}^d and let f be its homogeneous part of degree n , then

$$\lim_{k \rightarrow \infty} \frac{\sum_{(k,x) \in \Gamma} F(x)}{k^{n+q}} = \int_{\Delta} f(x) dx$$

where dx is the Lebesgue measure on Δ .

Let X be spherical of dimension n . We can apply this result in a straightforward way using the description of the ring $R(X, \mathcal{L})$ for \mathcal{L} an ample line bundle. Indeed, Equation (1) and Equation (2) give

$$\dim H^0(X, \mathcal{L}^{\otimes k}) = \sum_{(k,\lambda) \in \Gamma(X, \mathcal{L})} \dim V(\lambda) \sim \text{vol}(\Delta_{str}) k^{\dim \Delta_{str}}.$$

The function $\dim V(\lambda)$ is a polynomial function on the characters. It is given by the formula $F(\lambda) = \prod_{\alpha > 0} \frac{\langle \alpha, \rho + \lambda \rangle}{\langle \alpha, \rho \rangle}$. Set $f(\lambda) = \prod_{\alpha > 0} \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \rho \rangle}$.

Corollary 4.3.17. *We have $\deg \mathcal{L} = n! \text{vol}(\Delta_{str}) = n! \int_{\Delta(X, \mathcal{L})} f(x) dx$.*

4.3.4. *Smoothness criterion.* The same type of arguments were used by Brion [27] to give a smoothness criterion for spherical varieties. Since this is a local problem, we may assume that X is simple with closed orbit Y . The main idea is to compute the multiplicity of the ideal of Y^{10} using an integral on a simplex. Let $\mathcal{C}_Y^\vee(X)$ be the cone associated to X and let $\mathcal{C}_Y(X)$ be its dual. Let (f_i) be generators of $\mathcal{C}_Y(X)$ and define

$$\mathcal{C}(Y) = \mathcal{C}_Y(X) \setminus \bigcap_i (f_i + \mathcal{C}_Y(X)).$$

Recall the definition of the function f above.

Theorem 4.3.18. *The multiplicity of the ideal of Y is given by*

$$c! \int_{\mathcal{C}(Y)} f(x) dx$$

where c is the codimension of Y in X .

Corollary 4.3.19. *The variety X is smooth if and only if the cone $\mathcal{C}_Y^\vee(X)$ is generated by a basis of \mathbb{X}^\vee and $\mathcal{C}(Y)$ is a simplex of volume $1/\text{codim}(Y)!$ for the form $f(x) dx$.*

4.3.5. *Intersection theory.* Kaveh explained in [74] how to describe the subring of the cohomology ring $H^*(X, \mathbb{R})$ generated by the Picard group $\text{Pic}(X)$ thanks to the formula for the degree of any line bundle. The main observation is the following result.

Theorem 4.3.20. *Let $A = \sum_{i=0}^n A_i$ be a commutative, finite dimensional graded k -algebra, generated in degree 1, satisfying Poincaré duality and such that $A_0 = A_n = k$.*

Let $(H_i)_{i \in [1, r]}$ be a basis of A_1 and define the polynomial P by $P(x_1, \dots, x_r) = (\sum_i x_i H_i)^n \in A_n$. Then the algebra A is isomorphic to $k[Y_1, \dots, Y_r]/I$ where I is the ideal

$$I = \left\{ f(Y_1, \dots, Y_r) / f\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}\right) \cdot P = 0 \right\}.$$

¹⁰The multiplicity of an ideal I in a local ring A is the highest term in k of the function $\text{lg}(A/I^n)$.

Let X be a spherical projective variety of dimension n and fix $(H_i)_{i \in [1, r]}$ a basis of $\text{Pic}(X)$. For \mathcal{L} an ample line bundle, write $\mathcal{L} = \sum_i x_i H_i$ and define $P(x_1, \dots, x_r) = \deg(X, \mathcal{L}) = n! \text{vol}(\Delta_{str})$.

Corollary 4.3.21. *Assume that the subalgebra of $H^*(X, \mathbb{R})$ generated by $\text{Pic}(R)$ satisfies Poincaré duality, then this algebra is isomorphic to $\mathbb{R}[Y_1, \dots, Y_r]/I$ where I is the ideal*

$$I = \left\{ f(Y_1, \dots, Y_r) / f\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}\right) \cdot P = 0 \right\}.$$

This result in particular generalises Borel's presentation of the cohomology ring of complete flag varieties [14] and the description by [129] of the cohomology ring of toric varieties.

4.3.6. Comments. Theorem 4.3.1 was initiated by Guillemin and Steinberg [62], [63] and proved in full generality in [82]. Proposition 4.3.4 was proved in a more general setting by Atiyah [7]. We follow the proof of Brion [23]. The results on the moment map are mainly taken from [23]. The proof of Theorem 4.3.6 can be found in [35, Proposition 5.3.2].

Our presentation of toric degenerations follows [3], [5] and [77]. In a recent preprint [75], Kaveh recovers the parametrisation of the dual canonical basis via Okounkov type valuations as in Example 4.3.8 and recovers this way the multiplicative property of the dual canonical basis. Note also that Alexeev and Brion constructed moduli spaces for spherical varieties and some of their deformations in [4]. See also the survey [37].

The formula in Corollary 4.3.17 for the degree was first proved in [24] and recovered using general Okounkov bodies techniques in [77]. The smoothness criterion appears in [27, Théorème and Corollaire 4.1].

The comparison between the polytope ring and Schubert calculus for homogeneous spaces of type A have been pursued in [80] and [81].

The connection between spherical varieties and symplectic geometry is illustrated by the solution of Delzant's conjecture. In [51], Delzant conjectured that, if K is a compact Lie group, the symplectomorphic class of a multiplicity-free Hamiltonian K -manifold (multiplicity-free means here that K has a dense orbit in the preimage of any coadjoint orbit by the moment map) is determined by the convex polytope $\mu(X) \cap \mathfrak{t}_{K,+}^\vee$ (obtained as the intersection of the image of the moment map with the dominant chamber) and by its general stabiliser (the stabiliser of a general point in $\mu^{-1}(\mu(X) \cap \mathfrak{t}_{K,+}^\vee)$). Knop [93] proved that this conjecture is equivalent to the fact that affine spherical varieties are determined by their weight monoid. This in turn was proved by Losev [103], [104]. Furthermore Knop describes in [93], building on a classification of smooth affine spherical varieties in [94], which moment polytopes and general stabilisers actually occur. We refer to [52], [105] and [93] for more details on this topic.

4.4. B -orbits. An interesting combinatorial object associated to any spherical variety X is the set $B(X)$ of B -orbit closures in X . As for homogeneous spaces, this set has two natural orderings: Bruhat and weak Bruhat orders.

Bruhat order is given by inclusion. The weak Bruhat order on $B(X)$ is defined as follows. The maximal elements for the weak order are the closures of the G -orbits. If $Y \in B(X)$ is not G -stable, then there exists a minimal parabolic subgroup P

containing B such that $PY \neq Y$. We say that P raises Y to PY and write $Y \preccurlyeq PY$. These are the covering relations of the weak Bruhat order.

4.4.1. *Graph $\Gamma(X)$.* For a covering relation $Y \preccurlyeq PY$, we may consider the proper morphism $\pi : P \times^B Y \rightarrow PY$.

Proposition 4.4.1. *One of the following occurs.*

- *Type U:* $PY = Y \cup Y'$ and π is birational.
- *Type N:* $PY = Y \cup Y'$ and π has degree 2.
- *Type T:* $PY = Y \cup Y' \cup Y''$ for $Y'' \in B(X)$ with $\dim Y'' = \dim Y$ and π is birational.

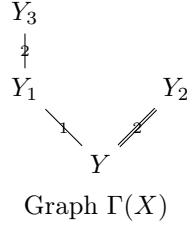
Proof. For x general in Y , write P_x for the stabiliser of x in P . If \mathcal{O} is the dense B -orbit in Y , the orbits in $P\mathcal{O}$ are in bijections with the orbits of P_x in P/B . Considering the image of P_x in $\text{Aut}(P/B)$ there are three cases: the image contains the maximal unipotent subgroup (type U), the image is the normaliser of a maximal torus (type N) or the image is a maximal torus (type T). \square

Definition 4.4.2. The graph $\Gamma(X)$ has $B(X)$ as set of vertices and has a P -edge of type U , T or N for each covering relation $Y \preccurlyeq PY$ of this type.

Example 4.4.3. For projective rational homogeneous spaces, the graph $\Gamma(X)$ is connected and has only edges of type U .

Some geometric properties of the orbit closures can be seen on the graph $\Gamma(X)$.

Corollary 4.4.4. *Assume that P_1 and P_2 raise Y to Y_1 and Y_2 with type U or T and N respectively and that P_2 raises Y_1 to Y_3 with type U or T , then Y_3 is not normal along Y_2 .*



- +— P_1 -edges of type U or T
- 2— P_2 -edges of type U or T
- == P_2 -edges of type N

Proof. The morphism $P_2 \times^B Y_1 \rightarrow Y_3$ is birational while its restriction $P_2 \times^B Y \rightarrow Y_2$ has non connected fibres. Zariski main Theorem gives the conclusion. \square

Example 4.4.5. Take $G = \text{Sp}_4$ and H the normaliser of the Levi subgroup of the maximal parabolic subgroup associated to the simple long root. Then G/H contains a configuration of B -orbits as in the above corollary (see [31, Example 1]).

4.4.2. *Normality criterion.* As Corollary 4.4.4 shows, in some cases, the existence of edges of type N in the graph $\Gamma(X)$ prevents some orbit closures from being normal. If there is no edge of type N , Brion proved the following general result.

Definition 4.4.6. A variety $Y \in B(X)$ is called multiplicity-free if there is no edge of type N in the full subgraph of $\Gamma(X)$ with vertices the elements $Y' \succcurlyeq Y$.

Theorem 4.4.7. *Let $Y \in B(X)$ be multiplicity-free and assume that for $Y' \succcurlyeq Y$, the variety Y' contains a G -orbit if and only if $Y' = GY$. If GY is normal, Cohen-Macaulay or has a rational resolution, then so does Y .*

Proof. We only sketch the proof of the normality and refer to [32] for the rest of the proof. We shall need the following two results.

For $Y \in B(X)$ and P raising Y write $\pi : P \times^B Y \rightarrow PY$. For any sheaf \mathcal{F} on $P \times^B Y$, we have $R^i \pi_* \mathcal{F} = 0$ for $i > 1$ and $R^1 \pi_* \mathcal{O}_{P \times^B Y} = 0$ (see for example [31, Page 294]).

For \mathcal{G} a sheaf on Y , write $P \times^B \mathcal{G}$ the corresponding sheaf induced on $P \times^B Y$. Then if \mathcal{G} is invariant under the stabiliser of Y in G and if $\pi_*(P \times^B \mathcal{F}) = 0$ for any P raising Y , then $\text{Supp } \mathcal{G}$ is G -invariant (see for example [31, Lemma 8]).

Assume that $G \cdot Y$ is normal. We prove by induction on the codimension in GY that Y is normal. Let $\nu : Z \rightarrow Y$ be the normalization and let \mathcal{F} be the cokernel of the map $\mathcal{O}_Y \rightarrow \nu_* \mathcal{O}_Z$. The sheaf \mathcal{F} is invariant under stabiliser of Y in G . The previous assertions imply that we have an exact sequence

$$0 \rightarrow \pi_* \mathcal{O}_{P \times^B Y} \rightarrow \pi_*(P \times^B \nu)_* \mathcal{O}_{P \times^B Z} \rightarrow \pi_*(P \times^B \mathcal{F}) \rightarrow 0$$

if P raises Y . But then $P \cdot Y$ is normal and both morphisms π , $\pi \circ (P \times^B \nu)$ are proper and birational. Thus, by Zariski's main Theorem the first two terms are isomorphic and $\pi_*(P \times^B \mathcal{F}) = 0$. We conclude that $\text{Supp } \mathcal{F}$ is G -invariant thus $\mathcal{F} = 0$ since Y contains no G -orbit. \square

Remark 4.4.8. In characteristic 0, the variety GY is always normal, Cohen-Macaulay and has a rational resolution.

The above result and proof is valid for any G -variety X , see [32]. This has a very nice application to subvarieties of homogeneous spaces that we explain in the next subsection.

4.4.3. Multiplicity-free classes. A natural geometric question for any (smooth) variety X and a homology class $\alpha \in H_*(X, \mathbb{Z})$ is to ask whether this class can be represented by a subvariety Y of X i.e. with $\alpha = [Y]$ (or more generally by a linear combination of homology classes of subvarieties). This is in general a very hard problem since the map $A_*(X) \rightarrow H_*(X)$ is not surjective. This does not occur for smooth spherical varieties, see Corollary 2.2.4. The following are also classical questions: which cohomology classes are represented by smooth subvarieties? More generally, given a cohomology class, what can we say on the varieties representing this class?

We focus now on the case where X is a projective rational homogeneous space. For $\alpha \in H_*(X, \mathbb{Z})$ a Schubert class, the problem of determining the (irreducible or smooth) subvarieties Y with $[Y] = \alpha$ is very classical (see [16, Section 5.17], [68], [65]). We only have partial answers. For curves this is completely solved in [122]. Very precise answers are given in the case of compact Hermitian spaces in [42], [70], [47], [136] and [135]. In particular, for many Schubert classes, the varieties representing these classes have to be singular.

More generally, it is natural to ask whether the cohomology class $[Y]$ of a subvariety Y of X imposes some geometric conditions on the variety. For examples,

codimension conditions on Y or intersections properties of $[Y]$ may impose geometric conditions on Y . (in particular simple connectedness results and results on the Picard group, see [6], [8], [9], [50], [55], [58], [64], [123], [124], [142].

The normality result of Theorem 4.4.7 implies the following striking result. A cohomology class $\alpha \in H^*(X, \mathbb{Z})$ is called multiplicity free if it is a non negative linear combination of Schubert classes with coefficients at most 1.

Theorem 4.4.9. *Any subvariety Y in X whose cohomology class is multiplicity-free is normal and Cohen-Macaulay. Furthermore for any nef line bundle \mathcal{L} on X the map $H^0(X, \mathcal{L}) \rightarrow H^0(Y, \mathcal{L})$ is surjective and $H^i(X, \mathcal{L}) = 0$ for $i > 0$.*

Proof. We deal with the case $X = G/B$ or more precisely $B \backslash G$ the quotient by the action by left multiplication. The general case is similar. Consider the quotient map $p : G \rightarrow X$, the variety $p^{-1}(Y)$ is B -stable (for the action of B on G by left multiplication). Let P be a minimal parabolic subgroup and consider $\pi_P : B \backslash G \rightarrow P \backslash G$. Then P raises $p^{-1}(Y)$ if and only if $Y \rightarrow \pi_P(Y)$ is generically finite. Furthermore the fibers of this map identifies with the fibers of the morphism $P \times^B p^{-1}(Y) \rightarrow p^{-1}(Y)$. Let d be their common degree and let α be a Schubert class such that $\alpha = \pi_P^* \beta$ with β a Schubert class in $P \backslash G$. Let α' be the unique Schubert class such that $\pi_{P*} \alpha' = \beta$. Projection formula gives

$$\int_X [Y] \cdot \alpha = d \int_X [\pi_P^{-1}(\pi_P(Y))] \cdot \alpha'$$

and the multiplicity-free assumption of $[Y]$ implies $d = 1$. In particular $p^{-1}(Y)$ is a multiplicity-free subvariety in the sense of spherical varieties (and G -varieties). Since G has a unique G -orbit, Theorem 4.4.9 implies that $p^{-1}(Y)$ and thus Y is normal and Cohen-Macaulay.

A similar argument implies the existence of flat deformations to a reduced Cohen-Macaulay union of Schubert varieties proving the cohomology statements. \square

4.4.4. Comments. Proposition 4.4.1 was first proved in [133] for symmetric varieties and extends readily to the general case. Brion studied the graph $\Gamma(X)$ in great details in [31]. The proof of Theorem 4.4.7 is taken from [32].

The geometry of B -orbit closures in spherical varieties is far from being completely understood. It has its origin in the importance of the geometry of Schubert varieties in projective rational homogeneous space. A combinatorial description of $\Gamma(X)$ as well as of the weak and strong orders for symmetric varieties was achieved in [133] and [134].

F. Knop in [91] defines an action of the Weyl group W of G on $B(X)$ and recovers the little Weyl group (see Subsection 3.4.1) as a subgroup of the stabiliser in W of X seen as an element of $B(X)$. N. Ressayre gives in [131] invariants characterising the orbits of the Weyl group action on $\Gamma(X)$. In [132] he gives a classification and several equivalent definitions of the spherical varieties for which W acts transitively on the B -orbits within the same G -orbit. These varieties are called spherical varieties of minimal rank and are also characterised by the fact that $\Gamma(X)$ has only edges of type U .

In general it is not known when the closures of B -orbits are normal. Normality and stronger results (such as globally F-regularity) have been proved by He and Thomsen [66] for closures of $B \times B$ -orbits in equivariant compactifications of

reductive groups (see also [67] for more results). However, even for symmetric varieties the closures of B -orbits are not normal in general (see Example 4.4.5). The normality of B -orbit closures is proved for some products of projective rational homogeneous spaces for a simply-laced semisimple group G which are spherical in [1].

5. EXAMPLES

In this section we give some examples of spherical varieties.

5.1. Toric varieties. Recall that a toric variety is a normal variety X with a dense orbit of a torus T . Toric varieties are spherical.

5.2. Projective rational homogeneous spaces. For G reductive and P a parabolic subgroup, the quotient G/P is a projective rational homogeneous space. Bruhat decomposition implies that they are spherical varieties.

5.3. Horospherical varieties. Horospherical varieties form the simplest class of spherical varieties containing both toric varieties and projective rational homogeneous spaces. Horospherical varieties are the embeddings of homogeneous spaces G/H such that H contains a maximal unipotent subgroup U of G . Bruhat decomposition implies their sphericity. Another characterisation of horospherical varieties is the equality $\mathcal{V} = \mathbb{X}^\vee$ (see for example [86, Corollary 6.2]). This simplifies the combinatorics involved in the classification by colored cones. For example, the toroidal horospherical varieties are the locally trivial fibrations $G \times^P Y$ over a projective rational homogeneous space G/P with fiber a variety Y toric for a quotient of a maximal torus T of P (see [119] for more precise results on horospherical varieties).

5.4. Spherical representations. Another very natural class of examples can be obtained by looking at rational representations of the group G . In particular, a complete classification of spherical representations has been obtained in [72] (for irreducible representations) and in [99] and [10] in general.

5.5. Products of homogeneous spaces. It is a classical problem to ask which product of projective rational homogeneous spaces $\prod_i G/P_i$ has a dense G -orbit. This is solved in [128] if all the parabolic subgroup agree and for G of type A or C in [109] and [110]. The more restricted problem of describing the products of projective rational homogeneous spaces $\prod_i G/P_i$ which are spherical is solved. Indeed, using the Structure Theorem 2.3.2 this problem reduces to the classification of spherical representations. The case of maximal parabolic subgroups was solved in [100], the general case can be found in [143, Corollaries 1.3.A-G]. Note that the case of complexity 1 has recently been solved in [126].

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